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MATHEMATICS MAGAZINE

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MATHEMATICS MAGAZINE

ROY DUBISCH, *EDITOR*

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PTOLEMY'S INEQUALITY AND THE CHORDAL METRIC

TOM M. APOSTOL, California Institute of Technology

1. Introduction. Claudius Ptolemy, the celebrated mathematician, astronomer and geographer who flourished in Alexandria during the 2nd century A.D., is best known for his *Almagest*, a remarkable treatise on astronomy consisting of thirteen books. In Book I, Ptolemy calculated a table of chords which is equivalent to a five-place table of sines from 0 to 90 degrees, at intervals of quarter degrees. His calculations are based on a lemma, now known as Ptolemy's Theorem, which may be stated as follows:

PTOLEMY'S THEOREM. *The product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the products of the lengths of the opposite sides.*

In other words, if the vertices are labeled in cyclic order as a, b, c, d, we have

$$ac \cdot bd = ab \cdot cd + bc \cdot ad \tag{1}$$

where ab, cd, etc., denote distances. This formula contains, as special cases, the Pythagorean Theorem, the addition formulas for the sine and cosine, and the half-angle formula $2 \sin^2(x/2) = 1 - \cos x$, all of which can be obtained by specializing the quadrilateral. (See [2], p. 83.)

An English translation of Ptolemy's simple proof of (1) is given in [4], p. 225. A different proof, based on inversion with respect to a circle through one of the vertices, may be found in [3], p. 157, and in [6], p. 64. Inversion also provides a proof of the following extension of Ptolemy's Theorem to arbitrary convex quadrilaterals.

PTOLEMY'S INEQUALITY. *If abcd is a convex quadrilateral, then we have*

$$ac \cdot bd \leqq ab \cdot cd + bc \cdot ad \tag{2}$$

with equality if and only if the quadrilateral is inscribed in a circle.

I. J. Schoenberg [7] has shown that Ptolemy's inequality holds if a, b, c, d are any four points in a real inner-product space. Metric spaces in which Ptolemy's inequality (2) holds for all points a, b, c, d are called *ptolemaic*. In general, the triangle inequality for the metric neither implies nor is implied by Ptolemy's inequality. Schoenberg [8] also proved that every real seminormed space which is ptolemaic must arise from a real inner-product space.

In this note we show that Ptolemy's inequality in the plane is an immediate consequence of the triangle inequality for complex numbers. Then we show that the inequality in the plane implies the inequality in 3-space. Finally, we prove that the three-dimensional Ptolemy inequality is equivalent to the triangle inequality for the chordal metric of complex-variable theory.

2. Ptolemy's inequality deduced from the triangle inequality. Let a, b, c, d be any four complex numbers. Applying the triangle inequality to the algebraic identity

$$(a - b)(c - d) + (b - c)(a - d) = (a - c)(b - d) \tag{3}$$

we immediately obtain inequality (2). Equation (3) shows that the moduli of the three complex numbers

$$(4)\qquad (a-b)(c-d), \qquad (b-c)(a-d), \qquad (a-c)(b-d)$$

are the lengths of the sides of a triangle. Therefore, we have equality in (2) if and only if the ratio

$$(a-b)(c-d)/(a-c)(b-d)$$

is real. But this ratio (the cross-ratio of a, b, c, d) is real if and only if a, b, c, d lie on a circle (see [1], p. 31). This proves the extended Ptolemy Theorem in the plane.

3. Ptolemy's inequality in 3-space. Now consider four noncoplanar points a, b, c, d in 3-space forming the vertices of a tetrahedron. We shall prove that we have the strict inequality

$$(5)\qquad ab\cdot cd + bc\cdot ad > ac\cdot bd.$$

In other words, *the products of the lengths of the three pairs of opposite edges of a tetrahedron always form the lengths of the sides of a triangle.*

In the figure, imagine the edge bd as a taut flexible string, with the remaining edges of the tetrahedron being rigid. Rotate vertex d about the axis ac until it lies in the plane of the base abc at, say, d', choosing the direction of rotation so that $bd' > bd$. Applying Ptolemy's inequality (2) to the quadrilateral $abcd'$ and noting that $cd = cd'$, $ad = ad'$, we find

$$ab\cdot cd + bc\cdot ad \geqq ac\cdot bd' > ac\cdot bd,$$

which proves (5).

4. Ptolemy's inequality and the chordal metric. The chordal distance $\chi(a, b)$ between two complex numbers a and b (see [1], p. 81, or [5], p. 42) is given by the equation

$$(6)\qquad \chi(a, b) = \frac{|a-b|}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}}\,.$$

In this section we prove that when a, b, c are three noncollinear points in the complex plane, the triangle inequality for the chordal metric,

$$(7)\qquad \chi(a, b) < \chi(a, c) + \chi(c, b),$$

is equivalent to the tetrahedral theorem discussed in the foregoing section. Using (6) we see that (7) is equivalent to the inequality

$$(8)\qquad |a-b|\sqrt{1+|c|^2} < |a-c|\sqrt{1+|b|^2} + |c-b|\sqrt{1+|a|^2}.$$

Construct a tetrahedron using as vertices the three points a, b, c in the complex plane and a fourth point d located at a distance 1 above the origin of the complex plane. The edges of this tetrahedron in the complex plane have lengths $|a-b|$, $|a-c|$, and $|c-b|$. The other three edges meeting at d have lengths

$\sqrt{1+|c|^2}$, $\sqrt{1+|b|^2}$, and $\sqrt{1+|a|^2}$. Therefore we see at once that Ptolemy's tetrahedral inequality implies (8). Conversely, inequality (8) implies Ptolemy's tetrahedral inequality for a tetrahedron with altitude 1. This, in turn, implies Ptolemy's inequality for a general tetrahedron.

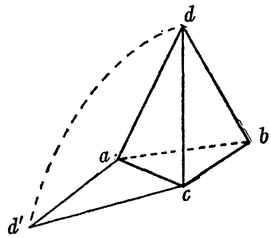

FIG. 1

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PASCAL-TYPE TRIANGLES FOR THE FOURIER EXPANSIONS OF $2^{n-1}\cos^n x$ AND $2^{n-1}\sin^n x$

BERNARD RASOF, Illinois Institute of Technology and Kabul University, Afghanistan

For positive integral n, the well known Pascal Triangle is a useful way of exhibiting coefficients of products of x and y in the expansion of $(x+y)^n$, and also provides a simple rule for computing these in terms of coefficients in the expansion of $(x+y)^{n-1}$, $n\geqq 1$. Here, after a short introduction to the Pascal Triangle, for the purpose of showing the similarities to, and differences from, our results, we present two new triangles, both of the Pascal type, which provide almost equally simple rules for rapid and easy computation of coefficients in the Fourier expansions of $2^{n-1}\cos^n x$ and $2^{n-1}\sin^n x$ in terms of coefficients in the Fourier expansions of $2^{n-2}\cos^{n-1} x$ and $2^{n-2}\sin^{n-1} x$, respectively. The procedures for entering numbers in the triangles are not entirely obvious, so they are derived for the Fourier expansion of $2^{n-1}\cos^n x$.

$\sqrt{1+|c|^2}$, $\sqrt{1+|b|^2}$, and $\sqrt{1+|a|^2}$. Therefore we see at once that Ptolemy's tetrahedral inequality implies (8). Conversely, inequality (8) implies Ptolemy's tetrahedral inequality for a tetrahedron with altitude 1. This, in turn, implies Ptolemy's inequality for a general tetrahedron.

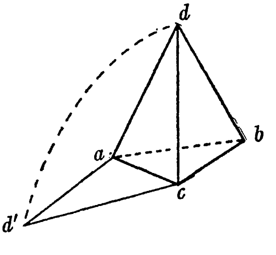

FIG. 1

References

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3. Howard Eves, A Survey of Geometry, vol. I, Allyn and Bacon, Boston, 1963.

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PASCAL-TYPE TRIANGLES FOR THE FOURIER EXPANSIONS OF $2^{n-1}\cos^n x$ AND $2^{n-1}\sin^n x$

BERNARD RASOF, Illinois Institute of Technology and Kabul University, Afghanistan

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n	x^n	$x^{n-1}y$	$x^{n-2}y^2$	$x^{n-3}y^3$	$x^{n-4}y^4$	$x^{n-5}y^5$	$x^{n-6}y^6$	$\cdots$	Sum of Coefficients
0	1								2^0
1	1	1							2^1
2	1	2	1						2^2
3	1	3	3	1					2^3
4	1	4	6	4	1				2^4
5	1	5	10	10	5	1			2^5
6	1	6	15	20	15	6	1		2^3
·	1	·	·	·	·	·	·	·	·

FIG. 1. The Pascal triangle for $(x+y)^n$.

1. Construction of the Pascal Triangle. For nonnegative integral n, in the expansion of $(x+y)^n$ coefficients of products of powers of x and y are conveniently displayed as a Pascal Triangle; one form is shown as Figure 1. There, under the columns headed by x^n, $x^{n-1}yx^{n-2}y^2$, $x^{n-3}y^3$, $\cdots$, for given n the entry on the corresponding row is the coefficient of that term in the expansion of $(x+y)^n$.

To construct Figure 1, place 1's in the column headed by x^n and along the diagonal leading from the uppermost 1. Numbers in the *interior* of the triangle are filled in starting at the top and working downwards; each interior number on row $n+1$ is the sum of the two numbers above and directly to its left on row n. On row n the sum of the coefficients is 2^n. These results are easily derived by use of mathematical induction.

2. Fourier expansions of $\cos^n x$ **and** $\sin^n x$**.** When n is a nonnegative integer the Fourier expansions of both $\cos^n x$ and $\sin^n x$ terminate, and each has two different forms, depending upon whether n is odd or even. Independently of n, because $\cos^n x$ is an even function, the Fourier expansion of $\cos^n x$ contains only a finite number of cosines of nonnegative integral multiples of x, through $\cos nx$. When the Fourier expansions of $2^0 \cos x$, $2 \cos^2 x$, $2^2 \cos^3 x$, $\cdots$ are written on successive lines, the coefficients of 1, $\cos x$, $\cos 2x$, $\cos 3x$, $\cdots$ fall into a pattern somewhat like the Pascal Triangle, and with certain strikingly similar characteristics. On the other hand, the Fourier expansion of $\sin^n x$ depends upon the nature of n. When n is odd, $\sin^n x$ is an odd function and its Fourier expansion contains only sines of positive integral multiples of x through $\sin nx$; for n even, $\sin^n x$ is an even function and its Fourier expansion consists of cosines of nonnegative integral multiples of x through $\cos nx$. When the Fourier expansions of $2^0 \sin x$, $2 \sin^2 x$, $2^2 \sin^3 x$, $\cdots$ are written one below the other, the coefficients of 1, sin x, cos $2x$, sin $3x$, cos $4x$, $\cdots$ also fall into a pattern somewhat like the triangle representing the Fourier expansion of $2^{n-1}\cos^n x$, but with a number of marked differences.

3. The Fourier Cosine Triangle for $2^{n-1}\cos^n x$**.** The representation of $\cos^n x$ entirely in terms of cosines of nonnegative integral multiples of x is actually its Fourier Cosine Expansion (FCE). The Pascal type FC triangle for $2^{n-1}\cos^n x$ is displayed as Figure 2. There, numbers on row n under the columns headed by 1, cos x, cos $2x$, $\cdots$ are the coefficients of those terms in the FCE of $2^{n-1}\cos^n x$.

n	$2^{n-1}\cos^n x$	1	cos x	cos $2x$	cos $3x$	cos $4x$	cos $5x$	cos $6x$	cos $7x$	cos $8x$	$\cdots$	Sum of Coefficients
1	$2^0 \cos x$	0	1									2^0
2	$2^1 \cos^2 x$	1	0	1								2^1
3	$2^2 \cos^3 x$	0	3	0	1							2^2
4	$2^3 \cos^4 x$	3	0	4	0	1						2^3
5	$2^4 \cos^5 x$	0	10	0	5	0	1					2^4
6	$2^5 \cos^6 x$	10	0	15	0	6	0	1				2^5
7	$2^6 \cos^7 x$	0	35	0	21	0	7	0	1			2^6
8	$2^7 \cos^8 x$	35	0	56	0	28	0	8	0	1		2^7
·	·	·	·	·	·	·	·	·	·	·		·

FIG. 2. The Fourier cosine triangle for $2^{n-1}\cos {}^n x$.

To construct Figure 2, (A) place 1's along the diagonal leading from the 1 at the head of the column of constant terms (in the uppermost row); (B) on a row for which n is odd, the entry in the column of constant terms is 0; (C) for n even, this entry is the coefficient of cos x in the FCE of $2^{n-2}\cos^{n-1} x$ (i.e., the number to the immediate right but on row $(n-1)$); (D) except for the column headed by cos x, in the interior of the triangle each entry on row n is the sum of the numbers directly to its left and its right on row $n-1$; (E) in the column headed by cos x, the entry on row n is the sum of the numbers directly to its right and twice that directly to its left on row $n-1$; (F) on row n the sum of the coefficients is 2^{n-1}.

4. Construction of the FC Triangle for $2^{n-1}\cos^n x$**.** Each of the statements (A)–(F) in the preceding paragraph is proved here. The proofs are slightly complicated by the circumstance that $2^{n-1}\cos^n x$ has two forms, depending upon whether n is odd or even.

For *any* nonnegative integral n,

$$
\begin{aligned}
2^n \cos^n x &= (e^{ix}+e^{-ix})^n = \binom{n}{0} e^{inx} + \binom{n}{1} e^{i(n-1)x}e^{-ix} \\
(1) \qquad &+ \binom{n}{2} e^{i(n-2)x}e^{-2ix} + \cdots + \binom{n}{n-2} e^{i2x}e^{-i(n-2)x} \\
&+ \binom{n}{n-1} e^{ix}e^{-i(n-1)x} + \binom{n}{n} e^{-inx},
\end{aligned}
$$

where $i=\sqrt{(-1)}$ and $\binom{n}{k}=n!/((n-k)!k!)=\binom{n}{n-k}$ are the binomial coefficients, which appear in the Pascal Triangle as the coefficients of $x^{n-k}y^k$ and x^ky^{n-k}, respectively. Thus, in (1), $\binom{n}{k}e^{i(n-k)x}e^{-ikx}=\binom{n}{k}e^{i(n-2k)x}$ and $\binom{n}{n-k}e^{ikx}e^{-i(n-k)x}=\binom{n}{n-k}e^{-i(n-2k)x}$ have equal coefficients, and can be combined to give $2\binom{n}{k}\cos(n-2k)x$. The number of terms in (1) is $n+1$, so when n is even one term has no mate; this is the single constant term, $\binom{n}{n/2}$. Then for n even, say $n=2m$, m a positive integer,

$$
(2) \qquad 2^{n-1}\cos^n x = 2^{2m-1}\cos^{2m} x = \frac{1}{2}\binom{2m}{m} + \sum_{k=1}^{m}\binom{2m}{m-k}\cos 2kx;
$$

while for n odd, $n=2m+1$, m a nonnegative integer,

$$
(3) \qquad 2^{n-1}\cos^n x = 2^{2m}\cos^{2m+1} x = \sum_{k=0}^{m}\binom{2m+1}{m-k}\cos(2k+1)x.
$$

Proof of (A). On row n the number on the outside diagonal is the entry under the column headed by $\cos nx$, and from both (2) and (3) the coefficient of $\cos nx$ is $\binom{n}{0}=1$.

Proof of (B). When n is odd, (3) shows that the FCE of $2^{n-1}\cos^n x$ begins with $\cos x$, so the constant term is 0.

Proof of (C). Consider $n-1$ odd; then $n-1=2m+1$, m a nonnegative integer. The coefficients of the FCE of $2^{n-1}\cos^n x$ are given by (2)—i.e., the entries on the odd row, $n-1=2m+1$ of Figure 2. To obtain entries on the even row $n=2m+2$—i.e., the coefficients in the FCE of $2^n\cos^{n+1}x$—multiply (3) by $2\cos x$, and into the result set the identity

$$
(4) \qquad \cos px\cos qx = \tfrac{1}{2}[\cos(p-q)x+\cos(p+q)x],
$$

yielding

$$
\begin{aligned}
2^n\cos^{n+1}x &= 2^{2m+1}\cos^{2m+2}x = \binom{2m+1}{m} + \sum_{k=1}^{m}\left\{\left[\binom{2m+1}{m-k}\right.\right. \\
(5) \qquad &\left.\left.+\binom{2m+1}{m-k+1}\right](\cos 2kx)\right\} + \binom{2m+1}{0}\cos 2(m+1)x.
\end{aligned}
$$

From (5), the constant term in the FCE of $2^n\cos^{n+1}x$ is the same number as the entry under the column headed by $\cos x$ in Figure 2 on row $2m+1$.

Proof of (D). When n is odd, put $n=2m+1$, m a nonnegative integer. In

Figure 2, entries on row n are the coefficients of the FCE of $2^{n-1}\cos^n x$, given by (3). To obtain entries on row $n+1$, multiply (3) by $2\cos x$, obtaining

$$(6)\quad 2^n \cos^{n+1} x = 2^{2m+2}\cos^{2m+2} x = \frac{1}{2}\binom{2m+2}{m+1} + \sum_{k=1}^{m+1}\binom{2m+2}{m-k+1}\cos 2kx.$$

From (6), on row $2m+2$ in the interior of Figure 2, the coefficient of $\cos 2kx$ is $\binom{2m+2}{m-k+1}$; from (3), on row $2m+1$ the coefficient of $\cos(2k-1)x$ is $\binom{2m+1}{m-k+1}$ and the coefficient of $\cos(2k+1)x$ is $\binom{2m+1}{m-k}$. The binomial coefficients add according to the law

$$(7)\qquad \binom{p}{q} + \binom{p}{q-1} = \binom{p+1}{q},$$

so that (D) follows immediately for n odd. When n is even, put $n=2m$, m a positive integer. Once more, given the coefficients of the FCE of $2^{n-1}\cos^n x$ from (2), we construct the coefficients of the FCE of $2^n \cos^{n+1} x$ by multiplying (2) by $2\cos x$, giving

$$(8)\quad \begin{aligned} 2^n\cos^{n+1} x = 2^{2m+1}\cos^{2m+1} x &= \binom{2m+1}{m-1}\cos x \\ &+ \sum_{k=1}^{m}\binom{2m+1}{m-k}\cos(2k+1)x \end{aligned}$$

after using (7). From (8), on row $2m+1$ in the interior of Figure 2 the coefficient of $\cos(2k+1)x$ is $\binom{2m+1}{m-k}$; in (3), the coefficient of $\cos 2kx$ is $\binom{2m}{m-k}$ and the coefficient of $\cos 2(k+1)x$ is $\binom{2m}{m-k-1}$; using (7), the sum of the latter two coefficients equals the former.

Proof of (E). Trivial for n even; to go from an even to odd row, take n even: $n=2m$, m a positive integer. The coefficient of $\cos x$ in the FCE of $2^n\cos^{n+1} x$ on row $(2m+1)$ is the first term in (8); with (7), $\binom{2m+1}{m-1} = \binom{2m}{m-1} + \binom{2m}{m}$, in which the right hand member is, in the FCE of $2^{n-1}\cos^n x$, the sum of the coefficient of $\cos 2x$ and twice the constant term.

Finally, to prove (F). From (1) the coefficients of $2^n\cos^n x$ are the binomial coefficients, with sum 2^n, so the FCE of $2^{n-1}\cos^n x$ has coefficients with sum 2^{n-1}.

5. Construction of the Fourier triangle for $2^{n-1}\sin^n x$. The Pascal type triangle for the Fourier expansion of $2^{n-1}\sin^n x$ is somewhat more complicated than that for the FCE of $2^{n-1}\cos^n x$, and is displayed as Figure 3. There, given n, numbers on row n under the columns headed by 1, $\sin x$, $\cos 2x$, $\sin 3x$, $\cos 4x, \cdots$ are the coefficients of those terms in the Fourier expansion of $2^{n-1}\sin^n x$.

Figure 3 is constructed as follows:

(A) All of the numbers on the diagonal have *absolute value* 1, beginning with $+1$, changing to -1 for the next two consecutive entries, returning to $+1$ for the following two consecutive entries, then back to -1 for the two succeeding entries, etc., in this way alternating in sign in groups of two.

(B) On row $n+1$, the entry in the column of constants (headed by 1) is the coefficient of $\sin x$ on the row n above.

(C) Under the column headed by $\sin x$ the entry on row $n+1$ is twice the entry on row n under the column headed by 1 less the entry under the column headed by $\cos 2x$.

(D) In the interior of the triangle, except for the entries under the column headed by $\sin x$, every entry on row $n+1$ is the sum of the absolute values of the two numbers directly to its left and to its right on row n, or the negative of this sum; the sign is that of the 1 on the diagonal intersecting the column. Alternatively, on row $n+1$ the entry under a column headed by a sine is the number directly to its left less that directly to its right, both on row n; the entry under a column headed by a cosine is the number directly to its right less that directly to its left, both also on row n.

(E) On row n the sum of the *absolute values* of the coefficients is 2^{n-1}.

Certain other features of the triangle of Figure 3 are of interest.

(F) In each column, every nonzero entry has the sign of the 1 on the diagonal intersecting that column.

(G) On each row, the nonzero numbers alternate in sign, always beginning with a positive number.

Proofs of the statements (A)–(G) above, explaining the construction of the triangle of Figure 3, are similar to those presented earlier to support the construction of Figure 2.

n	$2^{n-1}\sin^n x$	1	$\sin x$	$\cos 2x$	$\sin 3x$	$\cos 4x$	$\sin 5x$	$\cos 6x$	$\sin 7x$	$\cos 8x$	$\sin 9x$	$\cdots$	Sum of Absolute Values of Coefficients
1	$2^0 \sin x$	0	1										2^0
2	$2^1 \sin^2 x$	1	0	−1									2^1
3	$2^2 \sin^3 x$	0	3	0	−1								2^2
4	$2^3 \sin^4 x$	3	0	−4	0	1							2^3
5	$2^4 \sin^5 x$	0	10	0	−5	0	1						2^4
6	$2^5 \sin^6 x$	10	0	−15	0	6	0	−1					2^5
7	$2^6 \sin^7 x$	0	35	0	−21	0	7	0	−1				2^6
8	$2^7 \sin^8 x$	35	0	−56	0	28	0	−8	0	1			2^7
9	$2^8 \sin^9 x$	0	126	0	−84	0	36	0	−9	0	1		2^8
·	·	·	·	·	·	·	·	·	·	·	·	·	·

FIG. 3. The Fourier sine/cosine triangle for $2^{n-1} \sin {}^n x$.

ON THE ORIGINAL MALFATTI PROBLEM

MICHAEL GOLDBERG, Washington, D. C.

1. Introduction. In 1803, Malfatti (1737–1807), of the University of Ferrara, proposed the following problem [1]:

Given a right triangular prism of any sort of material, such as marble, how shall three circular cylinders of the same height as the prism and of the greatest possible volume be related to one another in the prism and leave over the least possible amount of material?

This reduces to the plane problem of cutting three circles from a given triangle so that the sum of their areas is maximized.

Malfatti, and many others who considered the problem, assumed that the solution would be the three circles which are tangent to each other, while each circle is tangent to two sides of the triangle, as in Figure 1a. These circles have become known as the Malfatti circles. The construction of the Malfatti circles, and the derivation of their sizes, have been the subject of many elegant papers. A brief history of these is given by Eves [2], and a more extensive history is given by Lob and Richmond [3]. The solution by Schellbach is given by Dörrie [4].

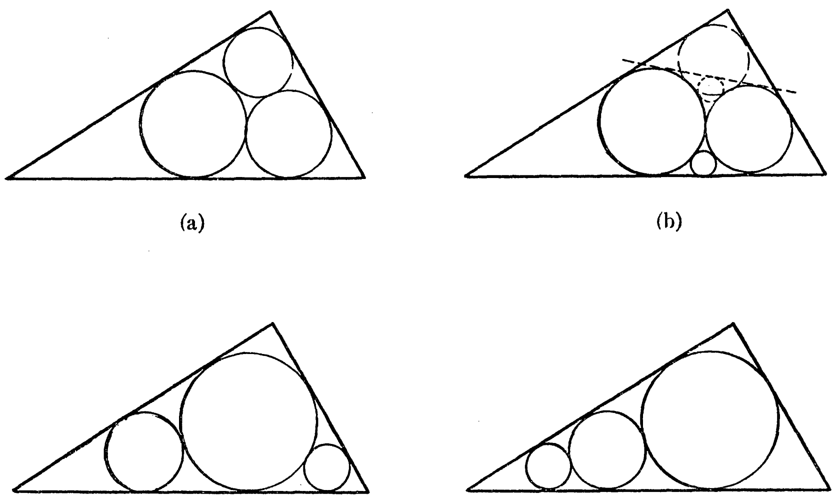

FIG. 1. Arrangements of circles.

It was not until 1929 that Lob and Richmond [3] noted that the Malfatti circles were not always the solution of the original Malfatti problem. In a brief note at the end of their paper, they remarked that for an equilateral triangle, the inscribed circle, with two little circles squeezed into the angles, contain a greater area than Malfatti's three circles. Eves [2] indicates that for very tall triangles, three circles placed one above the other can have a combined area greater than that of the Malfatti circles. It is the purpose of this note to show that the Malfatti circles are *never* the solution of the original Malfatti problem.

2. Cirles tangent to a line. A maximum area is not reached unless each circle is restrained from growing by making at least three contacts, either with the sides of the triangle or with other circles. These contacts must be distributed along the circumference so that they do not all lie within a semicircle; otherwise, the circle would not be in static equilibrium and could be enlarged by some adjustment. The Malfatti circles meet this condition and give a local maximum. However, other arrangements also meet this condition. The Malfatti arrangement is the only one in which each side touches only two circles, as shown in Figure 1a. In the other arrangements, the three circles touch the same side of the triangle. There are three such arrangements, shown in Figures 1b, 1c, and 1d. The middle circle may be the largest, the smallest, or the median circle. The case in which the smallest circle is in the middle, as shown in Figure 1b, can be improved by placing the smallest circle in the opposite angle where it can be larger, as shown in Figure 1a. The dotted line in Figure 1b is the other tangent to the two larger circles. By symmetry, the dotted circle is the same size as the smallest circle. However, by removing the constraint of the dotted line, the dotted circle can grow until it becomes the dashed circle when it touches one of the sides of the triangle. Each of the other cases may be best, depending upon the angles of the given triangle.

3. The Lob-Richmond-Goldberg construction (LRG). The following construction always yields a larger area than the Malfatti circles. First inscribe a circle in the given triangle. Then inscribe the second circle in the smallest angle and tangent to the first circle. The third circle may be inscribed in the same angle or in the next larger angle—whichever permits the larger circle. There are equivocal cases in which the two have the same area.

4. The radii of the Malfatti circles. The given triangles may have all possible shapes. Let us assume that they all have an inscribed circle of unit radius. Then, if the radii of the Malfatti circles are designated by r_1, r_2, r_3, it is shown by Lob and Richmond [**3**, p. 302] that

$$r_1 = (1+v)(1+w)/2(1+u), \qquad r_2 = (1+w)(1+u)/2(1+v),$$
$$r_3 = (1+u)(1+v)/2(1+w),$$

where $u = \tan A/4$, $v = \tan B/4$, $w = \tan C/4$, and A, B, C are the angles of the triangle.

If we maximize the sum of the squares of the radii, then this is equivalent to maximizing the area. For the Malfatti circles, let $M \equiv r_1^2 + r_2^2 + r_3^2$. Since $\tan\ (A/4+B/4+C/4) = \tan\ \pi/4 = 1$, we have $(u+v+w-uvw)/(1-vw-uw-uv) = 1$, from which $w = \{(1-uv)-(u+v)\}/\{(1-uv)+(u+v)\}$ and $1+w = 2(1-uv)/\{(1-uv)+(u+v)\}$. By means of this equation, the variable w can be eliminated, leaving M as a function of only u and v, namely:

$$M = (1+u)^2(1+v)^2(1+u+v-uv)^2/16(1-uv)^2$$
$$+ (1-uv)^2\{(1+u)^4+(1+v)^4\}/(1+u)^2(1+v)^2(1+u+v-uv)^2.$$

5. The radii of the LRG circles. The first circle has unit radius. Let the smallest angle be called A, and the next larger angle (or equal) be called B. Then, the radius of the second circle is given by $r_2=(1-\sin A/2)/(1+\sin A/2)$. If $\tan A/4=u$, then, since $\sin A/2=(2\tan A/4)/(1+\tan^2 A/4)=2u/(1+u^2)$, we have

$$r_2 = \{(1-u)/(1+u)\}^2,$$
$$r_3 = \{(1-v)/(1+v)\}^2, \quad \text{if the third circle is in } B, \quad (\textit{Case } 1).$$

or

$$r_3 = \{(1-u)/(1+u)\}^4, \quad \text{if the third circle is in } A, \quad (\textit{Case } 2).$$

Hence, $\text{LRG}(1)=r_1^2+r_2^2+r_3^2=1+\{(1-u)/(1+u)\}^4+\{(1-v)/(1+v)\}^4$, and $\text{LRG}(2)=1+\{(1-u)/(1+u)\}^4+\{(1-u)/(1+u)\}^8$.

6. The case of the equilateral triangle. The problem for the special case of the equilateral triangle has been extended by Procissi [5]. He inscribed a circle of radius y in one angle of an equilateral triangle of edge 2 and two circles of radius x in the other two angles, making the circles of radius x tangent to the circle of radius y. Then the relation between x and y is given by

$$y = \{2\sqrt{3} - x - \sqrt{8x(\sqrt{3}-x)}\}/3.$$

If S is the sum of the areas of the circles, then in Procissi's notation, $F(x) \equiv 9S/\pi = 9(2x^2+y^2)$. The graph of the function $F(x)$ is shown in Figure 2. The curve has a horizontal tangent near $x=0.27$. At this point, the function has the minimum value of 3.320.

The Malfatti circles are given by $x=y=(\sqrt{3}-1)/2=0.366$ for which $F(x)=3.618$. The LRG circles are given by $x=\sqrt{3}/9=0.192$, then $y\sqrt{3}/3 =0.577$, and $F(x)=3.667$. Each of these two values of $F(x)$ is a "corner maximum" since the slopes of the curve at these points are not zero.

For $x<0.192$, the circle of radius y protrudes outside of the triangle, and for $x>0.366$, the circles of radius x will overlap. Both of these cases are not admissible by the geometric conditions of the problem.

7. The general triangle. If a circle of radius x is inscribed in one angle of a given triangle and then circles of radii y and z are inscribed in the other angles, making the circles of radii y and z tangent to the circle of radius x, then y and z are expressible as functions of x. Therefore S is a function of x, say $F(x)=9S/\pi$. This function is similar to the function for the equilateral triangle; namely, it will have a minimum for some value of x, and two "corner maxima," one of which corresponds to the Malfatti circles and the other to three circles tangent to one side. In general, these curves can be made in three ways, depending upon the choice of the angle in which the circle of radius x is inscribed. It will be

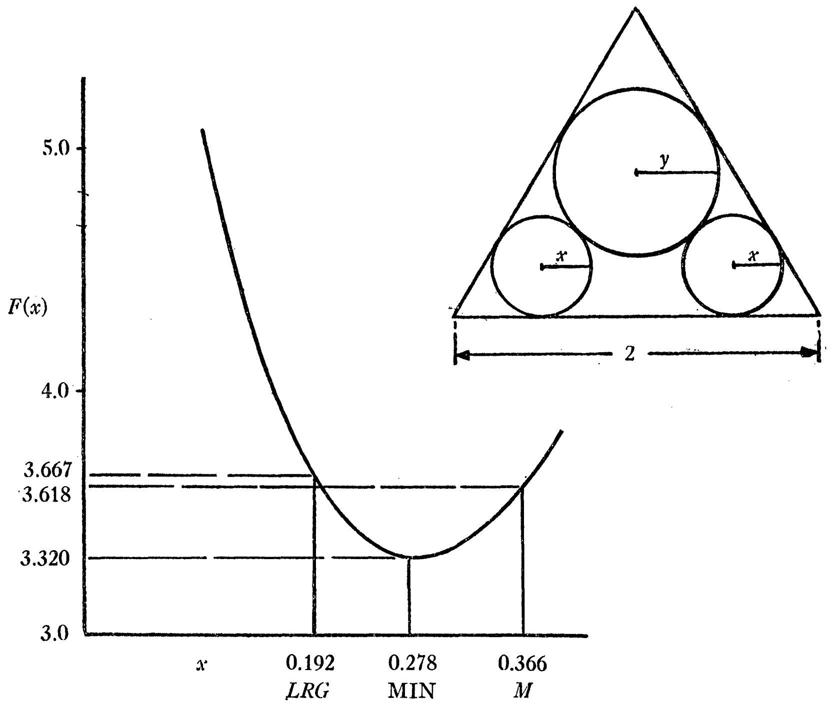

FIG. 2. Three circles in equilateral triangle.

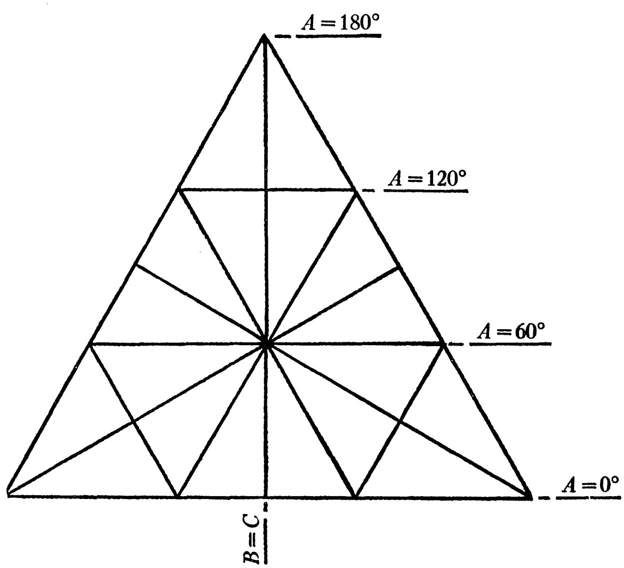

FIG. 3. Location of sections of surfaces.

shown that there is always a choice which makes the Malfatti sum the smaller of the two maxima.

8. Graphical demonstration of the nature of the surfaces. A rigorous algebraic proof of the greater values of the LRG sums over the M sums could become quite involved. It is proposed, therefore, to compute and describe the surfaces which represent the LRG and M sums as functions of the variables A, B, and C. Since $A+B+C=\pi$, we can indicate a triangle of angles A, B, and C as a point P in an equilateral triangle of height π (Figure 3). Then, the distances of the point from the sides of the triangle are A, B, and C. The continuous one-parameter family of isosceles triangles is represented by a median of the equilateral triangle. The values of LRG and M were computed for these isosceles triangles and are shown on the graph of Figure 4. Figure 5 shows the values for $A=0$, and $B+C=180°$. Figure 6 shows the values for $A=60°$ and $B+C=120°$. Figure 7 shows the values for $A=120°$ and $B+C=60°$. The curves on these graphs correspond to the sections of the surfaces cut by the planes indicated by the lines of Figure 3.

The M surface resembles a paraboloid of revolution. The LRG surface can be approximated by a segment of a paraboloid of revolution which has been deformed so that the circle at the top edge has been distorted into an equilateral triangle.

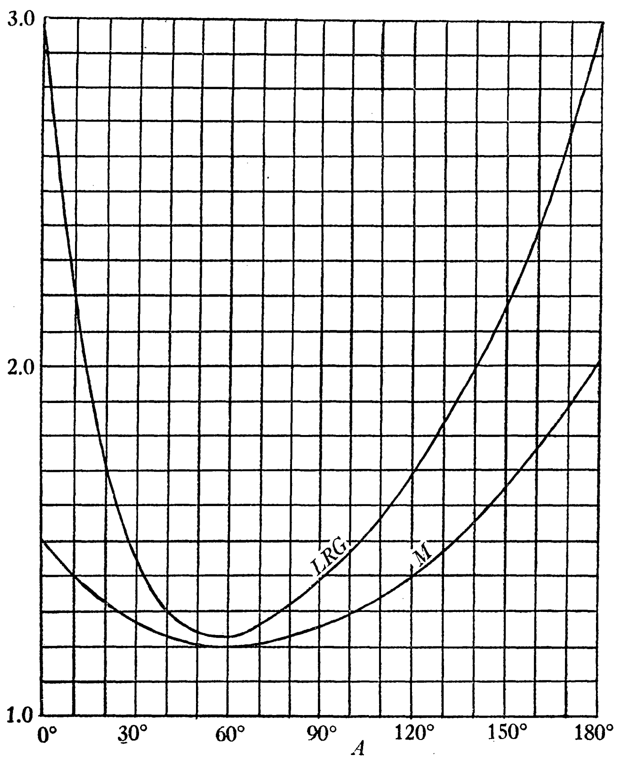

FIG. 4. Isosceles triangles, $B=C$.

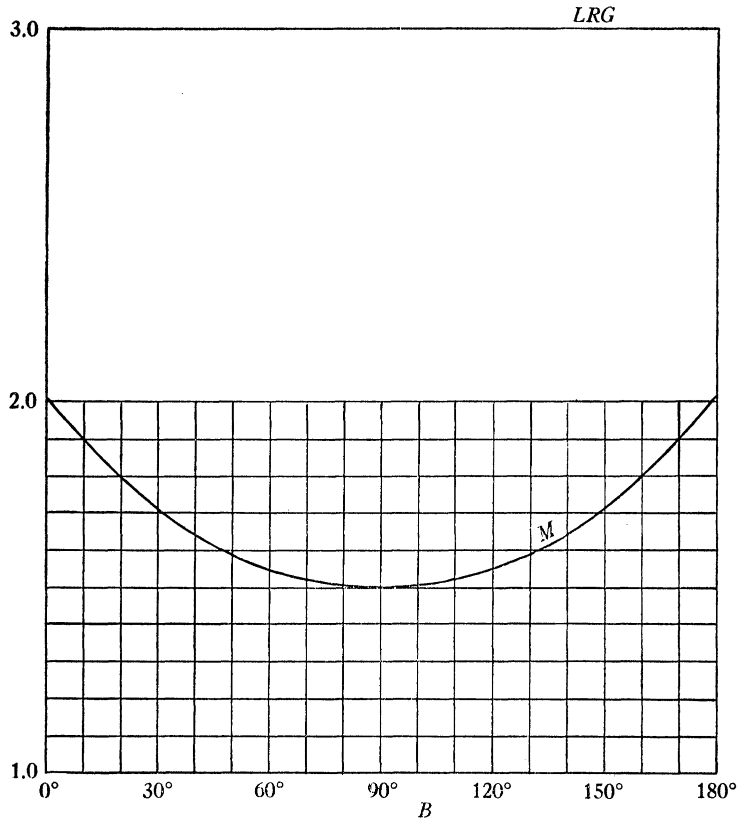

FIG. 5. $A=0°$.

For both surfaces, the lowest point occurs at $(A, B, C)=(60°, 60°, 60°)$. The axis of the surfaces is a vertical line through this point. A plane passing through this axis cuts the surfaces in two curves. One case is shown in Figure 4. Another case is shown in Figure 6. For other directions, an interpolated pair of curves is obtained. From the nature of the functions from which the surfaces are computed the curves are well behaved; they are continuous and have continuously increasing first derivatives. The surfaces made from these curves are, similarly, well behaved.

For each section through the axis, the curves have horizontal tangents at the axis. The ordinate of the lowest point of the M surface is $9(2-\sqrt{3})/2 = 1.206$. The ordinate of the lowest point of the LRG surface is slightly greater, namely, $11/9=1.222$. As we move away from the axis, the ordinates increase monotonically. The rate of increase on the LRG surface is always greater than the rate of increase on the M surface. Hence, over any point of the base triangle, the ordinate of the LRG surface is always greater than the ordinate of the M surface. This is evident from the numerical computation and graphing of the curves. A rigorous demonstration of this fact would be desirable, but it has not yet been developed.

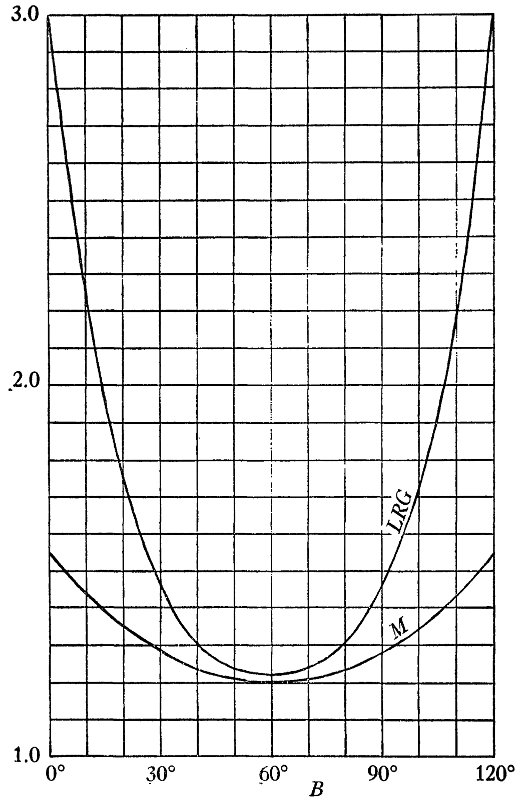

FIG. 6. $A=60°$.

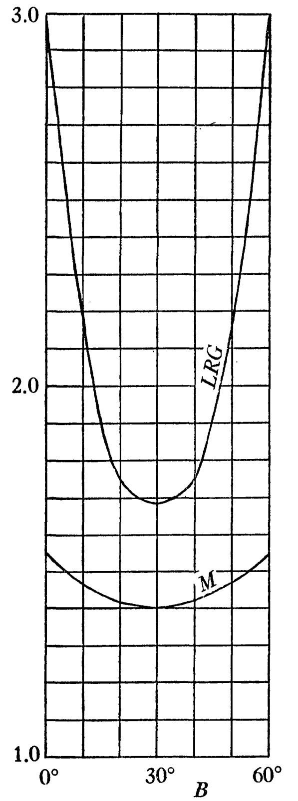

FIG. 7. $A=120°$.

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GENERALIZATIONS OF THE A.M. AND G.M. INEQUALITY

D. E. DAYKIN and C. J. ELIEZER, University of Malaya

Throughout this note Greek letters $\alpha, \beta, \cdots$ denote real numbers, Roman letters $a, b, \cdots$ denote positive real numbers, and capital Roman letters

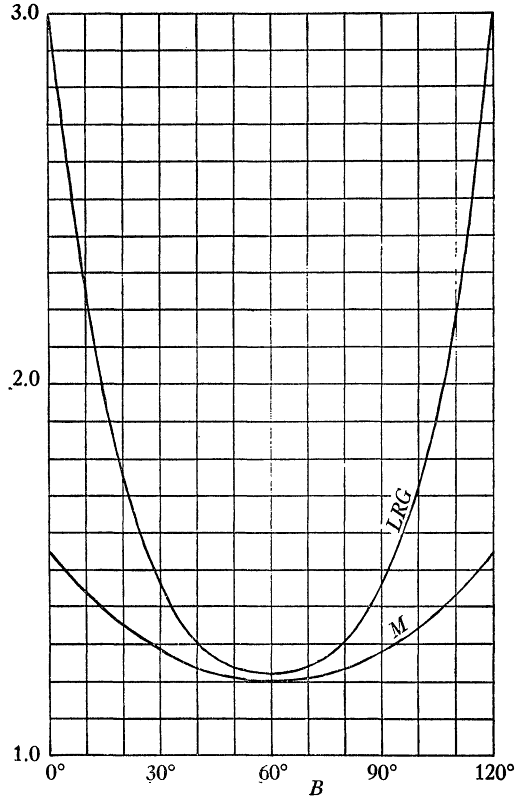

FIG. 6. $A=60°$.

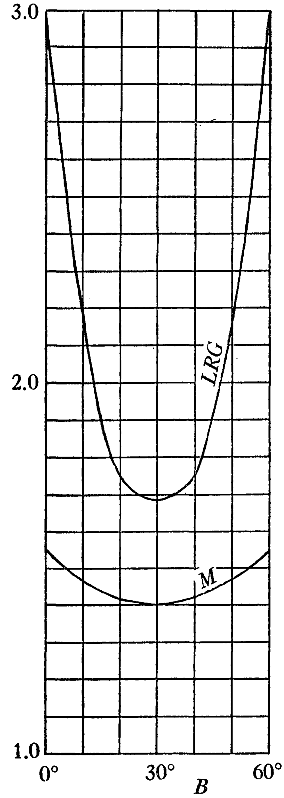

FIG. 7. $A=120°$.

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Throughout this note Greek letters $\alpha, \beta, \cdots$ denote real numbers, Roman letters $a, b, \cdots$ denote positive real numbers, and capital Roman letters

$M, N, \cdots$ denote positive integers or functions.

The Arithmetic Mean and Geometric Mean inequality is

$$\sqrt[M]{a_1 a_2 \cdots a_M} < (a_1 + a_2 + \cdots + a_M)/M \quad \text{unless } a_1 = a_2 = \cdots = a_M.$$

A familiar generalization of the inequality says that if $p_1+p_2+\cdots+p_M=1$ then

$$(1) \quad a_1^{p_1} a_2^{p_2} \cdots a_M^{p_M} < p_1 a_1 + p_2 a_2 + \cdots + p_M a_M \quad \text{unless} \quad a_1 = a_2 = \cdots = a_M.$$

When discussing inequalities it is interesting to look for a function which is convex and whose value increases smoothly from one side of the inequality to the other. Precisely what we mean by this statement will be clear from the sequel, and the kind of function we have in mind is given in

LEMMA 1. *The function*

$$F(x) = \sum_{I=1}^{M} a_I b_I^{\alpha_I x}$$

is strictly convex for all x, unless for each $I=1, 2, \cdots, M$ either $\alpha_I=0$ or $b_I=1$ when it is constant.

The result is trivial because

$$d^2F(x)/dx^2 = \sum ab^{\alpha x}(\alpha \log b)^2 \geqq 0.$$

Consider now the special case in which the function of the form $F(x)$ is

$$G(x) = (a_1^{p_1} \cdots a_M^{p_M})^x \sum_{I=1}^{M} p_I a_I^{1-(p_1+\cdots+p_M)x}.$$

Clearly

$$(2) \qquad G(0) = p_1 a_1 + p_2 a_2 + \cdots + p_M a_M,$$

and

$$(3) \qquad G(1) = a_1^{p_1} a_2^{p_2} \cdots a_M^{p_M} \quad \text{if } p_1 + p_2 + \cdots + p_M = 1,$$

so in view of the generalized A.M. and G.M. inequality (1) it is of interest to find the minimum of this convex function $G(x)$, and we do so in

THEOREM 1. *Let the minimum of $G(x)$ be attained at $x=x_0$; then $0<x_0<1$ or $x_0=1$ or $1<x_0$ according as $p=p_1+p_2+\cdots+p_M$ is >1 or $=1$ or <1.*

Equations (2) and (3) show that this theorem provides a wide generalization of inequality (1). An interesting result given by $G(0)<G(-1)$ is

$$(a_1^{p_1} \cdots a_M^{p_M})(p_1 a_1 + \cdots + p_M a_M) < p_1 a_1^{1+p} + \cdots + p_M a_M^{1+p}.$$

In order to prove Theorem 1, we need a result which is of interest on its own merits, namely

LEMMA 2. *Let*

$$P(\alpha) = (a_1^{p_1} \cdots a_M^{p_M})^{p_1 a_1^{\alpha} + \cdots + p_M a_M^{\alpha}}$$

and

$$Q(\alpha) = (a_1^{p_1 a_1^{\alpha}} \cdots a_M^{p_M a_M^{\alpha}})^{p_1 + \cdots + p_M},$$

then $P(\alpha) \lesseqqgtr Q(\alpha)$ *according as* $\alpha > 0$ *or* $\alpha = 0$ *or* $\alpha < 0$ *unless* $a_1 = a_2 = \cdots = a_M$.

In particular, putting $p_1 = p_2 = \cdots = p_M = \alpha = 1$ in Lemma 2 shows that $(a_1 \cdots a_M)^{a_1 + \cdots + a_M} < (a_1^{a_1} \cdots a_M^{a_M})^M$ unless $a_1 = a_2 = \cdots = a_M$.

To prove Lemma 2 we may assume $\alpha = 1$, for otherwise we simply make the substitution $a_I^{\alpha} = c_I$. We use induction on M. For the case $M = 2$ we note that

$$\frac{(a_1^{p_1} a_2^{p_2})^{p_1 a_1 + p_2 a_2}}{(a_1^{p_1 a_1} a_2^{p_2 a_2})^{p_1 + p_2}} = \left(\frac{a_1}{a_2}\right)^{p_1 p_2 (a_2 - a_1)} \leqq 1,$$

and for the general case we have

$$\frac{(a_1^{p_1} \cdots a_M^{p_M})^{p_1 a_1 + \cdots + p_M a_M}}{(a_1^{p_1 a_1} \cdots a_M^{p_M a_M})^{p_1 + \cdots + p_M}} = \left(\frac{a_1}{a_2}\right)^{p_1 p_2 (a_2 - a_1)} \cdots \left(\frac{a_1}{a_M}\right)^{p_1 p_M (a_M - a_1)}$$

$$\times \frac{(a_2^{p_2} \cdots a_M^{p_M})^{p_2 a_2 + \cdots + p_M a_M}}{(a_2^{p_2 a_2} \cdots a_M^{p_M a_M})^{p_2 + \cdots + p_M}},$$

from which the lemma follows.

To prove Theorem 1, by straightforward differentiation we find that

$$\frac{dG(0)}{dx} = \log \frac{P(1)}{Q(1)} < 0 \quad \text{by Lemma 2.}$$

Similarly

$$\frac{dG(1)}{dx} = a_1^{p_1} \cdots a_M^{p_M} \log \frac{P(\alpha)}{Q(\alpha)},$$

where α is given by $1 - \alpha = p_1 + p_2 + \cdots + p_M$, and the theorem follows by Lemma 2.

Another example of a function of the form of $F(x)$ is the convex function

$$H(x) = \sum_{I=1}^{M} p_I a_I (a_1 a_2 \cdots a_M \div a_I^M)^{p_1 p_2 \cdots p_M x / p_I},$$

for which

$$\frac{dH(0)}{dx} = p_1 p_2 \cdots p_M \log \frac{P(1)}{Q(1)} < 0,$$

where in $P(\alpha)$ and $Q(\alpha)$ alone we have $p_1=p_2=\cdots=p_M=1$. The inequality $H(0)<H(-1)$ is

$$\sum_{I=1}^{M} p_I a_I < \sum_{I=1}^{M} p_I a_I \left(\frac{\overset{M}{a_I}}{a_1 a_2 \cdots a_M}\right)^{p_1 p_2 \cdots p_M / p_I}.$$

SOME REMARKS ABOUT REAL ALMOST CONTINUOUS FUNCTIONS

TAQDIR HUSAIN, McMaster University

In view of the use of almost continuous mappings in [2], [3] and elsewhere, it is of some interest to compare properties of real almost continuous functions with those of real continuous functions.

A real-valued function f defined on the real line R or on a subset of it is said to be *almost continuous at* $x_0 \in R$ if, for each $\epsilon>0$, there exists a $\delta=\delta(\epsilon)>0$ such that the set $\{x: |f(x)-f(x_0)|<\epsilon\}$ is dense in the open interval $(x_0-\delta, x_0+\delta)$. In other words, the closure of the inverse image of each neighborhood of $f(x_0)$ is a neighborhood of x_0. f is almost continuous if it is so at each x_0.

It is clear that each continuous function is almost continuous. But the converse is not true, as follows from Example 2 at the end of this paper.

In this note we show among other results that a real almost continuous function on a closed bounded interval $[a, b]$ is not necessarily bounded (Example 5). Moreover, the image of a compact set under an almost continuous mapping need not be compact (Example 1). Thus almost continuous functions violate well known properties of continuous functions.

In contrast we have counterparts of two well known results for continuous functions. The uniform limit of a sequence of almost continuous functions is almost continuous (Theorem 2) and the pointwise limit of a sequence of such functions need not be almost continuous (Example 4).

We also give a number of examples by way of comparing almost continuous functions with semicontinuous functions, functions of Baire class one, Riemann integrable functions, and derivatives.

A real-valued function f defined on the real line R or on a subset of it is said to be *approximately continuous* at x_0 if, for each $\epsilon>0$, the set $G=\{x: |f(x)-f(x_0)|<\epsilon\}$ has the metric density 1 at x_0; i.e., for each interval I_δ of length δ with center x_0,

$$\text{Metric density} = \lim_{\delta\downarrow 0} \frac{m(G\cap I_\delta)}{m(I_\delta)} = 1,$$

where $m(A)$ denotes the Lebesgue measure of A.

Since the metric density of an open set at each of its points is equal to 1, it where in $P(\alpha)$ and $Q(\alpha)$ alone we have $p_1=p_2=\cdots=p_M=1$. The inequality $H(0)<H(-1)$ is

$$\sum_{I=1}^{M} p_I a_I < \sum_{I=1}^{M} p_I a_I \left(\frac{a_I^{M}}{a_1 a_2 \cdots a_M}\right)^{p_1p_2\cdots p_M/p_I}.$$

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where $m(A)$ denotes the Lebesgue measure of A.

Since the metric density of an open set at each of its points is equal to 1, it follows that each continuous function is approximately continuous. But the converse is not true.

First of all we prove a simple characterization of almost continuous functions similar to that for approximately continuous functions.

THEOREM 1. *A real-valued function f on R is almost continuous at x_0 if and only if there is a subset H of R which is dense in a neighborhood of x_0 and such that f is continuous at x_0 with respect to H; i.e., if $\{x_n\}$ is a sequence in H converging to x_0, then $\{f(x_n)\}$ converges to $f(x_0)$.*

Proof. The "if" part is obvious in view of the definition of almost continuity.

For the "only if" part, assume f is almost continuous at x_0. For each positive integer n there exists a $\delta_n > 0$ such that the set

$$H_n = \{x: |f(x) - f(x_0)| < 1/n\}$$

is dense in $(x_0 - \delta_n, x_0 + \delta_n)$. By induction, we can assume that $\{\delta_n\}$ is a strictly decreasing sequence of real numbers converging to 0. Since for each n, H_n is dense in $(x_0 - \delta_n, x_0 + \delta_n)$, $\bigcup_{n=1}^{\infty} H_n$ is dense in $(x_0 - \delta_1, x_0 + \delta_1)$. It is clear that $H_{n+1} \subset H_n$ for all $n \geqq 1$. Putting

$$G_n = H_n \cap \{(x_0 + \delta_{n+1}, x_0 + \delta_n) \cup (x_0 - \delta_n, x_0 - \delta_{n+1})\}$$

we observe that $G_n \cap G_m = \emptyset$ if $m \neq n$ and $H = \bigcup_{n=1}^{\infty} G_n \cup \{x_0\}$ is dense in $(x_0 - \delta_1, x_0 + \delta_1)$. Now to complete the proof, we show that f is continuous with respect to H. Let $\{x_m\}$ be a sequence in H converging to x_0. Then $x_m \in G_{n(m)}$ for each m and some $n(m)$, where $n(m)$ depends upon m. Hence $x_m \in H_{n(m)}$ and therefore $|f(x_m) - f(x_0)| < 1/(n(m))$. As $m \to \infty$, clearly $n(m) \to \infty$ and this proves that $\{f(x_m)\}$ converges to $f(x_0)$.

REMARK 1. A consequence of the above theorem is that each approximately continuous function is almost continuous, as pointed out in [**3**], Section 6.

REMARK 2. It is worth comparing Theorem 1 with a well known characterization of approximately continuous functions [**1**, pp. 312–313]. The set with respect to which an almost continuous function is continuous is required to be only dense, in contrast to the set with respect to which an approximately continuous function is continuous which is required to be of metric density 1. Thus it is easy to construct an almost continuous function which is not approximately continuous.

THEOREM 2. *Let $\{f_n\}$ be a sequence of almost continuous real functions on the real line R. If $\{f_n\}$ converges uniformly to a real function f, then f is almost continuous.*

Proof. Let $\epsilon > 0$ be given and let $x_0 \in R$. Since $\{f_n\}$ converges to f uniformly there exists n_0 such that for all $n \geqq n_0$, $|f_n(x) - f(x)| < \epsilon/3$ for all $x \in R$. For any fixed $n \geqq n_0$ we have $|f(x) - f(x_0)| \leqq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \leqq (2\epsilon/3) + |f_n(x) - f_n(x_0)|$.

Since f_n is almost continuous the set $G_n = \{x \in R: |f_n(x) - f_n(x_0)| < (\epsilon/3)\}$ is dense in some open interval $(x_0 - \delta, x_0 + \delta)$, $\delta > 0$. But clearly $G_n \subset \{x \in R:$

$|f(x)-f(x_0)|<\epsilon\}$. Hence the latter set is also dense in $(x_0-\delta, x_0+\delta)$. This proves that f is almost continuous at x_0.

REMARK. Theorem 2 compares with the well known result for continuous functions.

PROPOSITION 1. *Let f and g be two real-valued functions on R. If f is almost continuous and g continuous, then $f+g$ and fg are almost continuous.*

Proof. Let $\epsilon>0$ be given and let $x_0 \in R$. There exists a $\delta_1>0$ such that for all $x \in (x_0 - \delta_1, x_0 + \delta_1)$, $|g(x) - g(x_0)| < \epsilon/2$ and there exists $\delta_2 > 0$ such that $\{x: |f(x)-f(x_0)| < \epsilon/2\}$ is dense in $(x_0-\delta_2, x_0+\delta_2)$. Let $\delta=\min(\delta_1, \delta_2)$. Then $\{x: |f(x)-f(x_0)| < \epsilon/2\}$ is dense in $(x_0-\delta, x_0+\delta)$. But $(x_0-\delta, x_0+\delta) \cap \{x: |f(x)-f(x_0)| < \epsilon/2\}$ is a subset of $(x_0-\delta, x_0+\delta) \cap \{x: |f(x)+g(x)-f(x_0)-g(x_0)| < \epsilon\}$. This shows that the latter set is dense in $(x_0-\delta, x_0+\delta)$. Hence $f+g$ is almost continuous.

To show that fg is almost continuous, we consider $|f(x)g(x)-f(x_0)g(x_0)| \leqq |g(x)|\,|f(x)-f(x_0)|+|f(x_0)|\,|g(x)-g(x_0)|$. It can be assumed that $f(x_0)\neq 0$ without any loss of generality. Thus there exists a $\delta_1>0$ such that for all $x\in(x_0-\delta_1, x_0+\delta_1)$, $|g(x)-g(x_0)|<\epsilon/(2|f(x_0)|)$ by the continuity of g. Indeed g can be assumed to be bounded in $(x_0-\delta_1, x_0+\delta_1)$; i.e., $|g(x)| \leqq M$ for all $x\in(x_0-\delta_1, x_0+\delta_1)$. Now since f is almost continuous, there exists a δ_2 such that $\{x: |f(x)-f(x_0)| < \epsilon/(2M)\}$ is dense in $(x_0-\delta_2, x_0+\delta_2)$ and hence in $(x_0-\delta, x_0+\delta)$ where $\delta=\min(\delta_1, \delta_2)$. Now using the same arguments as above, we see that fg is almost continuous.

Another condition under which the above proposition is true is given by:

PROPOSITION 2. *Let f and g be two real almost continuous functions on R having the same set H with respect to which each f and g is continuous at each point of R. (See Theorem 1.) Then $f+g$ and fg are also almost continuous.*

Proof. Immediate.

PROPOSITION 3. *Let E, F, and G be three topological spaces. Let f be an almost continuous mapping of E into F and g a continuous mapping of F into G. Then the composition, $g \circ f$, is almost continuous.*

Proof. Let U be a neighborhood of $g(f(x)) \in G$ for $x \in E$. Then by the continuity of g, $g^{-1}(U)$ is a neighborhood of $f(x)$ and then by the almost continuity of f,

$$\overline{f^{-1}(g^{-1}(U))}$$

is a neighborhood of x. Hence $g \circ f$ is almost continuous.

REMARK. It is clear that each scalar multiple of an almost continuous function is almost continuous.

(A) *The image of a compact set under an almost continuous mapping need not be compact.*

Example 1. Let a real function f on $[0, 1]$ be defined as follows:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational in } [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

For each $\epsilon > 0$ and for any rational number $r_0 \in [0, 1]$, the set $G = \{x \in [0, 1]: |f(x) - r_0| < \epsilon\}$ consists of all rational numbers $x \in [0, 1]$ such that $r_0 - \epsilon < x < r_0 + \epsilon$, or all real numbers in $[0, 1]$ according as $\epsilon < r_0$ or $\epsilon \geqq r_0$. In both cases G is dense in a neighborhood of r_0. Hence f is almost continuous. However, the image of $[0, 1]$ under f is the set of all rational numbers in $[0, 1]$ which is noncompact.

(B) *An almost continuous function need not possess the intermediate-value property* (*i.e., if* $f(x_1) \leqq \alpha \leqq f(x_2)$, *for some real* α, *then there exists* x_3 *such that* $f(x_3) = \alpha$).

Example 2. The function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is almost continuous but does not possess the intermediate-value property.

(C) *An almost continuous function need not be a derivative.*

This follows from (B) and from the fact that each derivative satisfies the intermediate-value property ([**4**], p. 82, Theorem 5.12).

(D) *An almost continuous function need not be Riemann integrable. Nor need a Riemann integrable function be almost continuous.*

The function in Example 2, (B), is almost continuous but not Riemann integrable.

Example 3. Let f be defined on $[0, 1]$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leqq x < 1/2 \\ 1 & \text{if } 1/2 \leqq x \leqq 1. \end{cases}$$

Then it is easy to see that f is not almost continuous at $x = 1/2$. However, such a function being continuous almost everywhere (in this case outside the set consisting of $1/2$) is Riemann integrable.

(E) *A right* (*or left*) *semicontinuous function need not be almost continuous.*

This follows from Example 3.

(F) *A function of Baire class one* (*i.e., a discontinuous function which is the pointwise limit of a sequence of continuous functions*) *need not be almost continuous.*

Example 4. Let $f_n (n \geqq 1)$ and f be on $[-1, 1]$ as follows:

$$f_n(x) = \begin{cases} 1 & \text{if } 1/2 \leqq x \leqq 1 \\ n(x - 1/2) + 1 & \text{if } 1/2 - 1/n \leqq x \leqq 1/2 \\ 0 & \text{if } -1 \leqq x \leqq 1/2 - 1/n. \end{cases}$$

Each f_n is continuous. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Then it is easy to see that

$$f(x) = \begin{cases} 1 & \text{if } 1/2 \leqq x \leqq 1 \\ 0 & \text{if } -1 \leqq x < 1/2. \end{cases}$$

Clearly f is a function of Baire class one, but it is not almost continuous. (See Example 3.)

This example also shows that the pointwise limit of a sequence of almost continuous functions is not necessarily almost continuous.

(G) *An almost continuous function on* $[a, b]$ *need not be bounded.*

Example 5. Let $\{E_n\}$ be a sequence of pairwise disjoint sets in $[0, 1]$ such that each E_n is dense everywhere in $[0, 1]$ and $\bigcup E_n = [0, 1]$. It is not hard to construct such sets. Define $f(x) = n$ if $x \in E_n$ where $n = 1, 2, \cdots$. Then f is almost continuous but unbounded.

The author is thankful to Professor W. Orlicz for this example.

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Added in proof: Theorem 1 appeared recently in the Canad. Bull. of Math.

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Answers

A416.

$$I(t) = \int_0^1 \frac{dx}{(x^2+1)(x^t+1)} + \int_1^\infty \frac{dy}{(y^2+1)(y^t+1)} \cdot$$

In the second integral let $y = 1/x$. We then obtain

$$I(t) = \int_0^1 \frac{dx}{x^2+1} = \pi/4.$$

Thus the range of $I(t) = 0$. This integral appears in *Induction and Analogy in Mathematics*, by G. Polya.

A417. The relation can be verified by substituting $2r + x$ for R. When the triangle is equilateral, $x = 0$ and $R = 2r$.

A418. Including the two primes there are $2k+1$ consecutive integers which have a g.c.d. of $(2k+1)!$ Since the primes do not enter into this g.c.d., it must be the g.c.d. for the numbers between the primes.

A419. Put $S_1 = \{(P_1, P_2, \cdots, P_n) \mid 0 \leqq P_i \leqq 1, P_i \text{ rational}\}$, and

$$S_2 = \{(x_1, x_2, \cdots, x_n) \mid 2 < x < 3, 0 < x_1 < 1 \text{ for } i > 1\}.$$

These sets exhibit the characteristic indicated in the question.

(Quickies on page 289)

A PROBLEM ON THREE DESARGUEAN PAIRS OF TRIANGLES

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In the usual synthetic proof of Desargues' Theorem for two coplanar triangles we make use of Desargues' Theorem for two noncoplanar triangles. This gives rise to a plane figure [**1**, p. 31] or [2, p. 22] which shows that if each pair of three triangles is perspective from the same line, then they are perspective from three collinear points. The dual of this statement reads: "If each pair of three triangles is perspective from the same point, the three axes of perspectivity of these three triangles are concurrent." The proof for this is given as an exercise in many books [**3**, p. 154] and sometimes it is asked if the converse is also true. Evidently, the converse of such propositions is not necessarily true.

However, if we consider the coincident three points as collinear points and the coincident three lines as concurrent lines, we can put them together and attain a formulation which contains not only the above two statements, but also some other special cases. The following lines are devoted to the statement of the formulation and its analytic proof as a classroom note.

PROPOSITION. *Suppose every pair of three triangles* $A_1A_2A_3$, $B_1B_2B_3$, *and* $C_1C_2C_3$ *is Desarguean, and they do not have any vertex in common;* $A_1A_2A_3$ *and* $B_1B_2B_3$ *are perspective from a point* D *and a line* l; $A_1A_2A_3$ *and* $C_1C_2C_3$ *are perspective from a point* D' *and a line* l'; $B_1B_2B_3$ *and* $C_1C_2C_3$ *are perspective from a point* D'' *and a line* l''. *Then* D, D', *and* D'' *are collinear if and only if* l, l', *and* l'' *are concurrent.* (*Three coincident points are considered as collinear, and three coincident lines as concurrent.*)

Proof. From the assumption that these pairs are perspective respectively from D, D', and D'', we can choose coefficients of dependence of the points so that

$$\begin{aligned} &\lambda_1A_1+\mu_1B_1=\lambda_2A_2+\mu_2B_2=\lambda_3A_3+\mu_3B_3=D,\\ &\lambda_1'A_1+\nu_1C_1=\lambda_2'A_2+\nu_2C_2=\lambda_3'A_3+\nu_3C_3=D',\\ &\mu_1'B_1+\nu_1'C_1=\mu_2'B_2+\nu_2'C_2=\mu_3'B_3+\nu_3'C_3=D''.\end{aligned} \tag{1}$$

Moreover, each of the following three sets of points is collinear on the lines l, l', and l'' respectively where

$$\begin{aligned} &l:\lambda_1A_1-\lambda_2A_2=\mu_2B_2-\mu_1B_1, \qquad \lambda_2A_2-\lambda_3A_3=\mu_3B_3-\mu_2B_2,\\ &\quad\lambda_3A_3-\lambda_1A_1=\mu_1B_1-\mu_3B_3,\\ &l':\lambda_1'A_1-\lambda_2'A_2=\nu_2C_2-\nu_1C_1, \qquad \lambda_2'A_2-\lambda_3'A_3=\nu_3C_3-\nu_2C_2,\\ &\quad\lambda_3'A_3-\lambda_1'A_1=\nu_1C_1-\nu_3C_3.\\ &l'':\mu_1'B_1-\mu_2'B_2=\nu_2'C_2-\nu_1'C_1, \qquad \mu_2'B_2-\mu_3'B_3=\nu_3'C_3-\nu_2'C_2,\\ &\quad\mu_3'B_3-\mu_1'B_1=\nu_1'C_1-\nu_3'C_3.\end{aligned} \tag{2}$$

Suppose now that D, D', and D'' are collinear. Then there exist α, β, γ (at least two of them are nonzero) such that

$$\alpha D + \beta D' + \gamma D'' = 0. \tag{3}$$

From (1) and (3) we have

$$\begin{aligned} &(\alpha\lambda_1 + \beta\lambda_1')A_1 + (\alpha\mu_1 + \gamma\mu_1')B_1 + (\beta\nu_1 + \gamma\nu_1')C_1 = 0, \\ &(\alpha\lambda_2 + \beta\lambda_2')A_2 + (\alpha\mu_2 + \gamma\mu_2')B_2 + (\beta\nu_2 + \gamma\nu_2')C_2 = 0, \\ &(\alpha\lambda_3 + \beta\lambda_3')A_3 + (\alpha\mu_3 + \gamma\mu_3')B_3 + (\beta\nu_3 + \gamma\nu_3')C_3 = 0. \end{aligned} \tag{4}$$

Case I. If every triple of points (A_1, B_1, C_1), and (A_2, B_2, C_2), and (A_3, B_3, C_3) is independent, then

$$\begin{aligned} &\alpha\lambda_1 + \beta\lambda_1' = \alpha\mu_1 + \gamma\mu_1' = \beta\nu_1 + \gamma\nu_1' = 0, \\ &\alpha\lambda_2 + \beta\lambda_2' = \alpha\mu_2 + \gamma\mu_2' = \beta\nu_2 + \gamma\nu_2' = 0, \\ &\alpha\lambda_3 + \beta\lambda_3' = \alpha\mu_3 + \gamma\mu_3' = \beta\nu_3 + \gamma\nu_3' = 0. \end{aligned} \tag{5}$$

Since at least two of α, β, and γ are nonzero, it follows that

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1' & \lambda_2' & \lambda_3' \end{vmatrix}, \quad \begin{vmatrix} \mu_1 & \mu_2 & \mu_3 \\ \mu_1' & \mu_2' & \mu_3' \end{vmatrix}, \quad \begin{vmatrix} \nu_1 & \nu_2 & \nu_3 \\ \nu_1' & \nu_2' & \nu_3' \end{vmatrix}$$

are all of rank one. Thus we have, for example, $\lambda_1\lambda_2' - \lambda_2\lambda_1' = 0$, from which it follows that two points $\lambda_1 A_1 - \lambda_2 A_2$ and $\lambda_1' A_1 - \lambda_2' A_2$ are dependent; i.e., they are coincident points. We can see that two points $\lambda_1 A_1 - \lambda_2 A_2 = \mu_2 B_2 - \mu_1 B_1$ and $\mu_1' B_1 - \mu_2' B_2$ are coincident by the same kind of argument. Therefore these three points are coincident. Similarly, it can be shown that the three points $\lambda_2 A_2 - \lambda_3 A_3 = \mu_3 B_3 - \mu_2 B_2$, $\lambda_2' A_2 - \lambda_3' A_3$, and $\mu_2' B_2 - \mu_3' B_3$ are also coincident. Thus we have $l = l' = l''$.

Case II. (A_1, B_1, C_1) are collinear, but every triple of points (A_2, B_2, C_2) and (A_3, B_3, C_3) is independent. Then we have $(5)_2$ and $(5)_3$. Consequently

$$\begin{vmatrix} \lambda_2 & \lambda_2' \\ \lambda_3 & \lambda_3' \end{vmatrix} = 0, \qquad \begin{vmatrix} \mu_2 & \mu_2' \\ \mu_3 & \mu_3' \end{vmatrix} = 0, \qquad \begin{vmatrix} \nu_2 & \nu_2' \\ \nu_3 & \nu_3' \end{vmatrix} = 0. \tag{6}$$

In this case the three points $\lambda_2 A_2 - \lambda_3 A_3 = \mu_3 B_3 - \mu_2 B_2$, $\lambda_2' A_2 - \lambda_3' A_3 = \nu_3 C_3 - \nu_2 C_2$, and $\mu_2' B_2 - \mu_3' B_3 = \nu_3' C_3 - \nu_2' C_2$ coincide. Therefore l, l', and l'' meet at this point.

Case III. Both triples (A_1, B_1, C_1) and (A_2, B_2, C_2) are collinear. We have

$$\begin{aligned} D &= (A_1 \oplus B_1) \cap (A_2 \oplus B_2) = (A_1 \oplus B_1) \cap (A_3 \oplus B_3), \\ D' &= (A_1 \oplus C_1) \cap (A_2 \oplus C_2) = (A_1 \oplus C_1) \cap (A_3 \oplus C_3), \\ D'' &= (B_1 \oplus C_1) \cap (B_2 \oplus C_2) = (B_1 \oplus C_1) \cap (B_3 \oplus C_3), \end{aligned} \tag{7}$$

where $A_1 \oplus B_1$ means the line passing through points A_1 and B_1; $(A_1 \oplus B_1) \cap (A_2 \oplus B_2)$ means the intersection point of the two lines $A_1 \oplus B_1$ and $A_2 \oplus B_2$. Thus in this case we have $D = D' = D''$. Consequently (A_3, B_3, C_3) are also collinear.

III$_1$. If $\lambda_1 = \lambda_1'$, then it follows from $D = D' = D''$ and (1) that $\mu_1 B_1 - \nu_1 C_1 = 0$. As it is assumed that B_1 and C_1 do not coincide, it follows that $\mu_1 = \nu_1 = 0$. Then by (2), $l = B_2 \oplus B_3$, $l' = C_2 \oplus C_3$ and $(B_2 \oplus B_3) \cap (C_2 \oplus C_3) \in l''$, that is, l, l' and l'' are concurrent.

If $\lambda_1=\lambda_1'$ and $\lambda_2=\lambda_2'$, then, as above, we have $\mu_1=\nu_1=0$, $\mu_2=\nu_2=0$. Consequently, from $D=D'=D''$ and (1), we get $\lambda_1A_1=\lambda_2A_2$, which contradicts our assumption. Therefore $\lambda_1=\lambda_1'$ and $\lambda_2=\lambda_2'$ cannot happen simultaneously.

III$_2$. Suppose $\lambda_1\neq\lambda_1'$, $\lambda_2\neq\lambda_2'$ and $\lambda_3\neq\lambda_3'$. Then it follows from (1) that

$$(8)\qquad \begin{aligned} A_1 &= \frac{1}{\lambda_1-\lambda_1'}(\nu_1C_1-\mu_1B_1), \qquad A_2 = \frac{1}{\lambda_2-\lambda_2'}(\nu_2C_2-\mu_2B_2),\\ A_3 &= \frac{1}{\lambda_3-\lambda_3'}(\nu_3C_3-\mu_3B_3). \end{aligned}$$

Putting these expressions into (1), we get:

$$(9)\qquad \begin{aligned} D &= \frac{\lambda_1'\mu_1}{\lambda_1'-\lambda_1}B_1+\frac{\lambda_1\nu_1}{\lambda_1-\lambda_1'}C_1 = \frac{\lambda_2'\mu_2}{\lambda_2'-\lambda_2}B_2+\frac{\lambda_2\nu_2}{\lambda_2-\lambda_2'}C_2\\ &= \frac{\lambda_3'\mu_3}{\lambda_3'-\lambda_3}B_3+\frac{\lambda_3\nu_3}{\lambda_3-\lambda_3'}C_3. \end{aligned}$$

From (9) it is seen that the following three points are collinear on l'':

$$(10)\qquad \begin{aligned} \frac{\lambda_1'\mu_1}{\lambda_1'-\lambda_1}B_1-\frac{\lambda_2'\mu_2}{\lambda_2'-\lambda_2}B_2 &= \frac{\lambda_1\nu_1}{\lambda_1'-\lambda_1}C_1-\frac{\lambda_2\nu_2}{\lambda_2'-\lambda_2}C_2,\\ \frac{\lambda_2'\mu_2}{\lambda_2'-\lambda_2}B_2-\frac{\lambda_3'\mu_3}{\lambda_3'-\lambda_3}B_3 &= \frac{\lambda_2\nu_2}{\lambda_2'-\lambda_2}C_2-\frac{\lambda_3\nu_3}{\lambda_3'-\lambda_3}C_3,\\ \frac{\lambda_3'\mu_3}{\lambda_3'-\lambda_3}B_3-\frac{\lambda_1'\mu_1}{\lambda_1'-\lambda_1}B_1 &= \frac{\lambda_3\nu_3}{\lambda_3'-\lambda_3}C_3-\frac{\lambda_1\nu_1}{\lambda_1'-\lambda_1}C_1. \end{aligned}$$

III$_{2a}$. If $\lambda_1'(\lambda_2\lambda_3'-\lambda_2'\lambda_3)$ and $\lambda_3'(\lambda_1\lambda_2'-\lambda_2\lambda_1')$ do not vanish simultaneously, then

$$(11)\qquad \begin{aligned} E &= \{-\lambda_1'(\lambda_2\lambda_3'-\lambda_2'\lambda_3)(\lambda_1A_1-\lambda_2A_2)+\lambda_3'(\lambda_1\lambda_2'-\lambda_2\lambda_1')(\lambda_2A_2-\lambda_3A_3)\}\\ &= -\{\lambda_1'(\lambda_2\lambda_3'-\lambda_2'\lambda_3)\lambda_1A_1+\lambda_2'(\lambda_1'\lambda_3-\lambda_1\lambda_3')\lambda_2A_2\\ &\qquad +\lambda_3'(\lambda_1\lambda_2'-\lambda_2\lambda_1')\lambda_3A_3\}\\ &= \{-\lambda_1(\lambda_2\lambda_3'-\lambda_2'\lambda_3)(\lambda_1'A_1-\lambda_2'A_2)+\lambda_3(\lambda_1\lambda_2'-\lambda_2\lambda_1')(\lambda_2'A_2-\lambda_3'A_3)\}. \end{aligned}$$

Thus this point is the intersection of l and l'.

By (1), the point E can also be written in the following two ways:

$$(12)\qquad \begin{aligned} E &= \{-\lambda_1'(\lambda_2\lambda_3'-\lambda_2'\lambda_3)(\mu_2B_2-\mu_1B_1)+\lambda_3'(\lambda_1\lambda_2'-\lambda_2\lambda_1')(\mu_3B_3-\mu_2B_2)\}\\ &= (\lambda_1'-\lambda_1)(\lambda_2\lambda_3'-\lambda_2'\lambda_3)\left(\frac{\lambda_1'\mu_1}{\lambda_1'-\lambda_1}B_1-\frac{\lambda_2'\mu_2}{\lambda_2'-\lambda_2}B_2\right)\\ &\quad -(\lambda_3'-\lambda_3)(\lambda_1\lambda_2'-\lambda_1'\lambda_2)\left(\frac{\lambda_2'\mu_2}{\lambda_2'-\lambda_2}B_2-\frac{\lambda_3'\mu_3}{\lambda_3'-\lambda_3}B_3\right)\cdot \end{aligned}$$

This means that the point E is also the intersection point of l and l''. Thus it is shown that, in this case, l, l', and l'' are concurrent.

III_{2b}. Suppose that $\lambda_1'(\lambda_2\lambda_3'-\lambda_2'\lambda_3)$ and $\lambda_3'(\lambda_1\lambda_2'-\lambda_2\lambda_1')$ vanish simultaneously.

If $\lambda_1'=\lambda_1\lambda_2'-\lambda_2\lambda_1'=0$, it follows from $\lambda_1'-\lambda_1\neq 0$ that $\lambda_1'=\lambda_2'=0$. From $\lambda_3'=(\lambda_2\lambda_3'-\lambda_2'\lambda_3)=0$, it also follows that $\lambda_2'=\lambda_3'=0$. But all such cases as well as the case $\lambda_1'=\lambda_3'=0$ cannot happen because, for example, if $\lambda_1'=\lambda_2'=0$, we have $\nu_1C_1=\nu_2C_2=D$, which contradicts our assumption.

Thus, the only possible case is $(\lambda_2\lambda_3'-\lambda_2'\lambda_3)=(\lambda_1\lambda_2'-\lambda_2\lambda_1')=0$. Then $\lambda_1A_1-\lambda_2A_2=\mu_2B_2-\mu_1B_1$ and $\lambda_1'A_1-\lambda_2'A_2=\nu_2C_2-\nu_1C_1$ coincide. Furthermore, $\lambda_2A_2-\lambda_3A_3=\mu_3B_3-\mu_2B_2$ and $\lambda_2'A_2-\lambda_3'A_3=\nu_3C_3-\nu_2C_2$ also coincide. Consequently, $l=l'$ and l, l', l'' are concurrent.

In the above, it has been shown that if D, D', and D'' are collinear, then l, l', and l'' are concurrent. The converse follows from this result and the principle of duality.

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ON THE CONVERGENCE OF TAYLOR SERIES FOR FUNCTIONS OF n VARIABLES

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In this paper we shall prove a theorem, for functions of n variables, which is analogous to a theorem of S. N. Bernstein for functions of one variable: If F and all its derivatives are nonnegative in an interval I, then F is analytic in the interval. (See [1] and [2].) The two variable case is included in a more general theorem in [5]. [4] shows that if $(-1)^m\Delta^m f(x_1, \cdots, x_n)\geqq 0$, for every nonnegative integer m, in some open set A then there is an open set $U\subset A$ in which f is analytic. Here Δ^m is the iterated Laplace operator.

THEOREM. *If f and its partial derivatives are nonnegative in an open set A in E^n, then given any point x_0 in A there exists an open spherical neighborhood P about x_0 (with x_0 as center) so that the Taylor expansion of f about x_0 converges to f for every $x\in P$.*

Let the point x_0 have cartesian coordinates $(x_0^1, x_0^2, \cdots, x_0^n)$; then by definition of open set there exists a spherical neighborhood S of radius ρ about x_0 so that $S\subset A$. Let P be the open spherical region of radius $\rho-\epsilon$, $\epsilon>0$ with center at x_0. Define Γ to be the set $\{(x^1, x^2, x^3, \cdots, x^n) \mid x^i-x_0^i>0,\ 0\leqq i\leqq n,\ (x^1, x^2, \cdots, x^n)\in P\}$. We will show convergence of the Taylor series in Γ and then extend it to P.

Let $x\in\Gamma$. Taylor's formula for n variables (see [3]) is

This means that the point E is also the intersection point of l and l''. Thus it is shown that, in this case, l, l', and l'' are concurrent.

III_{2b}. Suppose that $\lambda_1'(\lambda_2\lambda_3' - \lambda_2'\lambda_3)$ and $\lambda_3'(\lambda_1\lambda_2' - \lambda_2\lambda_1')$ vanish simultaneously.

If $\lambda_1' = \lambda_1\lambda_2' - \lambda_2\lambda_1' = 0$, it follows from $\lambda_1' - \lambda_1 \neq 0$ that $\lambda_1' = \lambda_2' = 0$. From $\lambda_3' = (\lambda_2\lambda_3' - \lambda_2'\lambda_3) = 0$, it also follows that $\lambda_2' = \lambda_3' = 0$. But all such cases as well as the case $\lambda_1' = \lambda_3' = 0$ cannot happen because, for example, if $\lambda_1' = \lambda_2' = 0$, we have $\nu_1 C_1 = \nu_2 C_2 = D$, which contradicts our assumption.

Thus, the only possible case is $(\lambda_2\lambda_3' - \lambda_2'\lambda_3) = (\lambda_1\lambda_2' - \lambda_2\lambda_1') = 0$. Then $\lambda_1 A_1 - \lambda_2 A_2 = \mu_2 B_2 - \mu_1 B_1$ and $\lambda_1' A_1 - \lambda_2' A_2 = \nu_2 C_2 - \nu_1 C_1$ coincide. Furthermore, $\lambda_2 A_2 - \lambda_3 A_3 = \mu_3 B_3 - \mu_2 B_2$ and $\lambda_2' A_2 - \lambda_3' A_3 = \nu_3 C_3 - \nu_2 C_2$ also coincide. Consequently, $l = l'$ and l, l', l'' are concurrent.

In the above, it has been shown that if D, D', and D'' are collinear, then l, l', and l'' are concurrent. The converse follows from this result and the principle of duality.

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ON THE CONVERGENCE OF TAYLOR SERIES FOR FUNCTIONS OF n VARIABLES

JAMES THOMAS DAY, Mathematics Research Center, University of Wisconsin

In this paper we shall prove a theorem, for functions of n variables, which is analogous to a theorem of S. N. Bernstein for functions of one variable: If F and all its derivatives are nonnegative in an interval I, then F is analytic in the interval. (See [**1**] and [2].) The two variable case is included in a more general theorem in [**5**]. [**4**] shows that if $(-1)^m \Delta^m f(x_1, \cdots, x_n) \geqq 0$, for every nonnegative integer m, in some open set A then there is an open set $U \subset A$ in which f is analytic. Here Δ^m is the iterated Laplace operator.

THEOREM. *If f and its partial derivatives are nonnegative in an open set A in E^n, then given any point x_0 in A there exists an open spherical neighborhood P about x_0 (with x_0 as center) so that the Taylor expansion of f about x_0 converges to f for every $x \in P$.*

Let the point x_0 have cartesian coordinates $(x_0^1, x_0^2, \cdots, x_0^n)$; then by definition of open set there exists a spherical neighborhood S of radius ρ about x_0 so that $S \subset A$. Let P be the open spherical region of radius $\rho - \epsilon$, $\epsilon > 0$ with center at x_0. Define Γ to be the set $\{(x^1, x^2, x^3, \cdots, x^n) \mid x^i - x_0^i > 0, 0 \leqq i \leqq n, (x^1, x^2, \cdots, x^n) \in P\}$. We will show convergence of the Taylor series in Γ and then extend it to P.

Let $x \in \Gamma$. Taylor's formula for n variables (see [**3**]) is

$$f(x) = f(x_0) + \sum_{i=1}^{n} f_i(x_0)(x^i + x_0^i) + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}(x_0)(x^i - x_0^i)(x^j - x_0^j) + \cdots + \frac{1}{(q-1)!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_{q-1}=1}^{n} f_{i_1 i_2\cdots i_{q-1}}(x_0)h^{i_1}h^{i_2}\cdots h^{i_{q-1}} + R_q(x). \tag{1}$$

Here $f_i(x_0) = (\partial f(x_0)/\partial x_i)$, etc.; $\vec{h} = \vec{x} - \vec{x}_0$, $h^{i_j} = x^{i_j} - x_0^{i_j}$, etc. $R_q(x)$ is given by

$$R_q(x) = \frac{1}{(q-1)!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_q=1}^{n} h^{i_1}h^{i_2}\cdots h^{i_q} \int_0^1 (1-t)^{q-1} f_{i_1 i_2\cdots i_q}(\vec{x}_0 + t\vec{h})dt. \tag{2}$$

Consider a point $p \in \Gamma$ on the line which joins x_0 to x, chosen so that $x_p^i > x^i$ for all $1 \leqq i \leqq n$. Let $\vec{z} = \vec{p} - \vec{x}_0$, $z^i = x_p^i - x_0^i$. Let the parametric representation of the line joining x_0 and p be given by $\vec{x}_0 + \lambda\vec{z}$, $0 \leqq \lambda \leqq 1$. Since x is on this line there exists a λ_0, $0 < \lambda_0 < 1$ such that $\vec{x} = \vec{x}_0 + \lambda_0\vec{z}$. Thus (2) simplifies to read

$$R_q(x) = \frac{1}{(q-1)!}\lambda_0^q \sum_{i_1=1}^{n}\cdots\sum_{i_q=1}^{n} z^{i_1}z^{i_2}\cdots z^{i_q} \int_0^1 (1-t)^{q-1} f_{i_1 i_2\cdots i_q}(\vec{x}_0 + t\vec{h})dt. \tag{3}$$

Since f and all its partial derivatives are nonnegative in A we have that

$$f_{i_1 i_2\cdots i_q}(\vec{x}_0 + t\vec{h}) \leqq f_{i_1 i_2\cdots i_q})\vec{x}_0 + t\vec{z}) \quad \text{for } t \in [0, 1],$$

and all possible choices of subscripts $i_1, \cdots, i_q$. Thus we have that

$$0 \leqq R_q(x) \leqq \frac{1}{(q-1)!}\lambda_0^q \sum_{i_1=1}^{n}\cdots\sum_{i_q=1}^{n} z^{i_1}z^{i_2}\cdots z^{i_q} \int_0^1 (1-t)^{q-1} f_{i_1 i_2\cdots i_q}(\vec{x}_0 + t\vec{z})dt. \tag{4}$$

Inequality (4) in another form is

$$0 \leqq R_q(x) \leqq \lambda_0^q R_q(p). \tag{5}$$

Since f and all its partial derivatives are nonnegative in A, we have that $R_q(p) \leqq f(p)$. Thus we obtain the inequality

$$0 \leqq R_q(x) \leqq \lambda_0^q f(p). \tag{6}$$

Since $0 < \lambda_0 < 1$, we have that

$$\lim_{q\to\infty} R_q(x) \to 0.$$

The cases in which some of the $h^i=0$ can be treated in the same manner. Hence we have the convergence of the Taylor series to the function f on the set Ω, where $\Omega=\{x \mid x^i-x_0^i \geqq 0,\ x\in P\}$.

Consider a point $y\in P$, $y\notin\Omega$, with coordinates $(y^1, y^2, \cdots, y^n)$. Since Ω is spherical there is a point $x\in\Gamma$ so that $|y^i-x_0^i|=x^i-x_0^i$, $1\leqq i\leqq n$. The nonnegativity of f and all its partial derivatives gives us the inequality

$$0 \leqq f_{i_1 i_2 \cdots i_q}(\vec{x}_0 + t(\vec{y} - \vec{x}_0)) \leqq f_{i_1 i_2 \cdots i_q}(\vec{x}_0 + t(\vec{x} - \vec{x}_0)) \tag{7}$$

for $t\in[0, 1]$ and all choices of subscripts. It is easily seen that

$$0 \leqq |R_q(y)| \leqq R_q(x). \tag{8}$$

Since $x\in\Gamma$ we have that

$$\lim_{q\to\infty} R_q(y) \to 0.$$

This completes the proof of the theorem.

In a written communication Professor R. L. Boas has informed the author that he gave an elementary proof of the two dimensional case of this theorem in an undergraduate thesis at Harvard.

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GEOMETRIC CONJUGATE OF A POINT RELATIVE TO CENTRAL QUADRIC SURFACES

KRISHAN K. GOROWARA, Wright State University

1. In ordinary three dimensional Euclidean space consider the central quadric surface $S\equiv ax^2+by^2+cz^2-1=0$. The line

$$(x-\alpha)/l = (y-\beta)/m = (z-\gamma)/n = r, \tag{1}$$

passing through the point $A(\alpha, \beta, \gamma)$ and having direction numbers (l, m, n), will intersect the surface $S=0$ in two points whose distances from A are the roots of the equation $a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma+nr)^2=1$ or

$$r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0. \tag{2}$$

Let r_1, r_2 be the roots of (2). Consider a point R on the line given by (1). Let

The cases in which some of the $h^i=0$ can be treated in the same manner. Hence we have the convergence of the Taylor series to the function f on the set Ω, where $\Omega=\{x \mid x^i-x_0^i \geqq 0,\ x\in P\}$.

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$$(7) \qquad 0 \leqq f_{i_1 i_2 \cdots i_q}(\vec{x}_0 + t(\vec{y} - \vec{x}_0)) \leqq f_{i_1 i_2 \cdots i_q}(\vec{x}_0 + t(\vec{x} - \vec{x}_0))$$

for $t\in[0, 1]$ and all choices of subscripts. It is easily seen that

$$(8) \qquad 0 \leqq |R_q(y)| \leqq R_q(x).$$

Since $x\in\Gamma$ we have that

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passing through the point $A(\alpha, \beta, \gamma)$ and having direction numbers (l, m, n), will intersect the surface $S=0$ in two points whose distances from A are the roots of the equation $a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma+nr)^2=1$ or

$$(2) \qquad r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

Let r_1, r_2 be the roots of (2). Consider a point R on the line given by (1). Let this point R be at a distance ρ from A such that ρ is the geometric mean of r_1 and r_2. We define the point R to be the geometric conjugate of A wrt (with respect to) $S=0$. The locus of such points R, as (l, m, n) varies, we shall call the geometric conjugate surface of A wrt $S=0$. Since ρ is the geometric mean of r_1 and r_2, we have $\rho=\sqrt{(r_1r_2)}$, $\rho^2=r_1r_2=(a\alpha^2+b\beta^2+c\gamma^2-1)/(al^2+bm^2+cn^2)$ and

$$\rho^2(al^2 + bm^2 + cn^2) = a\alpha^2 + b\beta^2 + c\gamma^2 - 1. \tag{3}$$

To obtain the locus of R we replace ρl, ρm, ρn by $x-\alpha$, $y-\beta$, $z-\gamma$ respectively. Hence the locus of R is given by

$$a(x - \alpha)^2 + b(y - \beta)^2 + c(z - \gamma)^2 = a\alpha^2 + b\beta^2 + c\gamma^2 - 1, \tag{4}$$

which is a central quadric surface with center at the point $A(\alpha, \beta, \gamma)$ and the squares of the lengths of the semiaxes of this central quadric surface are

$$(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)/a, \quad (a\alpha^2 + b\beta^2 + c\gamma^2 - 1)/b, \quad (a\alpha^2 + b\beta^2 + c\gamma^2 - 1)/c.$$

An immediate consequence is that if the point (α, β, γ) is on the surface $S=0$, then the locus of R is

$$a(x - \alpha)^2 + b(y - \beta)^2 + c(z - \gamma)^2 = 0,$$

which is a cone with vertex (α, β, γ) such that direction numbers of the generators satisfy the equation $al^2+bm^2+cn^2=0$.

2. Simplifying equation (4) we get

$$G \equiv ax^2 + by^2 + cz^2 - 2a\alpha x - 2b\beta y - 2c\gamma z + 1 = 0. \tag{5}$$

Taking the intersection of the surfaces $G=0$ and $S=0$ we get

$$T \equiv a\alpha x + b\beta y + c\gamma z - 1 = 0, \tag{6}$$

which is the equation of the polar of the point (α, β, γ) wrt the surface $S=0$. Thus: the polar of a point (α, β, γ) wrt a given central quadric surface is the intersection of the given central quadric surface and the geometric conjugate surface of the point (α, β, γ) wrt the given central quadric surface.

A shorter form of the equation of the geometric conjugate of $A(\alpha, \beta, \gamma)$ wrt $S=0$ is

$$S - 2T = 0. \tag{7}$$

3. The equations of the geometric conjugate surfaces of the points

$$(\alpha_1, \beta_1, \gamma_1), \quad (\alpha_2, \beta_2, \gamma_2), \quad (\alpha_3, \beta_3, \gamma_3) \text{ wrt } S = 0$$

are, respectively,

$$G_1 \equiv ax^2 + by^2 + cz^2 - 2a\alpha_1 x - 2b\beta_1 y - 2c\gamma_1 z + 1 = 0, \tag{8}$$

$$G_2 \equiv ax^2 + by^2 + cz^2 - 2a\alpha_2 x - 2b\beta_2 y - 2c\gamma_2 z + 1 = 0, \tag{9}$$

$$G_3 \equiv ax^2 + by^2 + cz^2 - 2a\alpha_3 x - 2b\beta_3 y - 2c\gamma_3 z + 1 = 0. \tag{10}$$

The intersections of $G_1=0$, $G_2=0$, $G_3=0$ taken in pairs are given by

(11) $$G_1 = G_2\colon ax(\alpha_1 - \alpha_2) + by(\beta_1 - \beta_2) + cz(\gamma_1 - \gamma_2) = 0,$$

(12) $$G_2 = G_3\colon ax(\alpha_2 - \alpha_3) + by(\beta_2 - \beta_3) + cz(\gamma_2 - \gamma_3) = 0,$$

(13) $$G_3 = G_1\colon ax(\alpha_3 - \alpha_1) + by(\beta_3 - \beta_1) + cz(\gamma_3 - \gamma_1) = 0.$$

These equations, being of the first degree, are equations of planes. Each of these planes passes through the origin. We denote these planes by $G_{p12}=0$, $G_{p23}=0$, $G_{p31}=0$ respectively and call them the geometric conjugate planes of the pairs of points $[(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2)]$, $[(\alpha_2, \beta_2, \gamma_2), (\alpha_3, \beta_3, \gamma_3)]$, $[(\alpha_3, \beta_3, \gamma_3), (\alpha_1, \beta_1, \gamma_1)]$ respectively. The intersection of these planes taken in pairs is a straight line whose equation is $G_1=G_2=G_3$.

Considering the planes $G_{p12}=0$, $G_{p23}=0$, $G_{p31}=0$, we can easily see that if $G_{p12}=0$ passes through $(\alpha_3, \beta_3, \gamma_3)$ and if $G_{p23}=0$ passes through $(\alpha_1, \beta_1, \gamma_1)$, then $G_{p31}=0$ passes through $(\alpha_2, \beta_2, \gamma_2)$. This result has perfect symmetry.

4. Suppose that the loci $G_1=0$, $G_2=0$ pass through the points $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_1, \beta_1, \gamma_1)$ respectively. Then we have

$$a\alpha_2^2 + b\beta_2^2 + c\gamma_2^2 - 2a\alpha_1\alpha_2 - 2b\beta_1\beta_2 - 2c\gamma_1\gamma_2 + 1 = 0$$

and

$$a\alpha_1^2 + b\beta_1^2 + c\gamma_1^2 - 2a\alpha_1\alpha_2 - 2b\beta_1\beta_2 - 2c\gamma_1\gamma_2 + 1 = 0.$$

Subtracting the two equations we obtain $a\alpha_1^2+b\beta_1^2+c\gamma_1^2=a\alpha_2^2+b\beta_2^2+c\gamma_2^2$. Hence: if the geometric conjugate surface of a point $(\alpha_1, \beta_1, \gamma_1)$ passes through $(\alpha_2, \beta_2, \gamma_2)$ and that of $(\alpha_2, \beta_2, \gamma_2)$ passes through $(\alpha_1, \beta_1, \gamma_1)$, then

$$a\alpha_1^2 + b\beta_1^2 + c\gamma_1^2 = a\alpha_2^2 + b\beta_2^2 + c\gamma_2^2.$$

It is easily seen that this algebra is reversible.

5. Any point on the line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n=r$ has coordinates $(\alpha+lr, \beta+mr, \gamma+nr)$. The geometric conjugate surface of this point wrt $S=0$ is

$$ax^2 + by^2 + cz^2 - 2a(\alpha + lr)x - 2b(\beta + mr)y - 2c(\gamma + nr)z + 1 = 0$$

or

$$ax^2 + by^2 + cz^2 - 2a\alpha x - 2b\beta y - 2c\gamma z - 2r(alx + bmy + cnz) + 1 = 0.$$

Since the point P is any point on the given line, the above equation is true for all r. Hence

$$ax^2 + by^2 + cz^2 - 2a\alpha x - 2b\beta y - 2c\gamma z + 1 = 0, \quad alx + bmy + cnz = 0.$$

This locus we shall call the geometric conjugate of the given line wrt the central quadric surface $S=0$.

6. In Section 1 we took the distance, ρ, of R from A as the geometric mean of r_1 and r_2. Suppose now that ρ is the arithmetic mean of r_1 and r_2, i.e., $\rho=(r_1+r_2)/2$. Then $\rho=-(a\alpha l+b\beta m+c\gamma n)/(al^2+bm^2+cn^2)$ or

$$\rho^2(al^2 + bm^2 + cn^2) = -\rho(a\alpha l + b\beta m + c\gamma n).$$

Hence the locus of R (which we call the arithmetic conjugate of $A(\alpha, \beta, \gamma)$ wrt $S=0$) is

$$ax^2 + by^2 + cz^2 - a\alpha x - b\beta y - c\gamma z = 0,$$

which is again a central quadric surface passing through the origin and center at the point $(\alpha/2, \beta/2, \gamma/2)$.

The intersection of the arithmetic conjugate of A with $S=0$ is again the polar of A, namely $T=0$.

With the usual notation the equation of the arithmetic conjugate can be written as $S=T$.

Again, let $A_i \equiv ax^2+by^2+cz^2-a\alpha_i x-b\beta_i y-c\gamma_i z=0$ be the arithmetic conjugates of $(\alpha_i, \beta_i, \gamma_i)$ $(i=1, 2, 3)$. Then the intersections of the arithmetic conjugates taken in pairs are the same as the geometric conjugate plane of the pairs of points.

This work has been supported in part by Contract No. AF 33(615)-1915 while the author was at Wright-Patterson Air Force Base, Ohio, during summer, 1966.

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UNIFORM APPROXIMATION OF REAL CONTINUOUS FUNCTIONS ON THE REAL LINE BY INFINITELY DIFFERENTIABLE FUNCTIONS

LYLE E. PURSELL, Grinnell College and University of Missouri at Rolla

The various extensions of the Weierstrass Approximation Theorem given by M. H. Stone in [3] apply to bounded functions. In this paper we use a *partition of unity* to show that any real continuous function on the real line R can be uniformly approximated by a real, infinitely differentiable function on R. Hence if the set $C^\infty(R)$ of all real, infinitely differentiable functions on R is considered as an "extended" ($0 \leqq \text{distance} \leqq \infty$) metric space with the metric $d(f, g) = \sup |f-g|$, then the completion of $C^\infty(R)$ via Cauchy sequences may be identified with the space $C(R)$ of all real, continuous functions on R. At the end of this paper we outline how our method of constructing the approximating function can be extended to Euclidean spaces of higher dimension and other differentiable manifolds.

THEOREM. *If f is any real, continuous function defined on the real line R and $\epsilon > 0$, then there is a real, infinitely differentiable function f_ϵ defined on R such that $|f(x) - f_\epsilon(x)| < \epsilon$ for all x in R.*

LEMMA. *There exists a real, infinitely differentiable function, which we will call the bump function b, such that:*

Hence the locus of R (which we call the arithmetic conjugate of $A(\alpha, \beta, \gamma)$ wrt $S=0$) is

$$ax^2 + by^2 + cz^2 - a\alpha x - b\beta y - c\gamma z = 0,$$

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LEMMA. *There exists a real, infinitely differentiable function, which we will call the bump function b, such that:*

$$\text{(i)}\qquad \begin{aligned} b(0) &= 1 \\ 0 < b(x) &\leqq 1 \qquad \text{for } -1 < x < 1, \\ b(x) &= 0 \qquad \text{for } x \leqq -1 \quad \text{or} \quad 1 \leqq x; \end{aligned}$$

(ii) *For any integer n and positive integer k*

$$\sum_{i=n}^{n+k} b(x-i) = 1 \qquad \text{for } n \leqq x \leqq n+k.$$

Note. The set of functions, $b(x-i)$, $i=0, \pm 1, \pm 2, \cdots$, constitute an example of a *partition of unity.* (See [1], p. 8.)

Proof of Lemma. The function

$$\begin{aligned} e(x) &= \exp\left((x-1)^{-1} - (x+1)^{-1}\right) \qquad &&\text{for } -1 < x < 1 \\ &= 0 &&\text{for } x \leqq -1 \quad \text{or} \quad 1 \leqq x \end{aligned}$$

is infinitely differentiable on R and is positive for $-1<x<1$. (See [2], pp. 25–26 or [1], p. 3.) Let Z denote the set of all integers and consider the set of all functions of the form $e(x-n)$, n in Z. For any given n, there are only three functions, $e(x-n+1)$, $e(x-n)$, and $e(x-n-1)$, in this set which are not identically zero on the interval $n-1 \leqq x \leqq n+1$. Hence the double infinite series $\sum_{n\in Z} e(x-n)$ converges to an infinitely differentiable function $E(x)$, which is positive everywhere. Also

$$\begin{aligned} \sum_{i=n}^{n+k} e(x-i) &= E(x) \qquad &&\text{for } n \leqq x \leqq n+k \\ &= e(x-n) &&\text{for } n-1 < x < n \\ &= e(x-n-k) &&\text{for } n+k < x < n+k+1 \\ &= 0 &&\text{for } x \leqq n-1 \quad \text{or} \quad n+k+1 \leqq x. \end{aligned}$$

One can easily show that the function $b(x) = e(x)/E(x)$ has the desired properties.

Proof of Theorem. By the Weierstrass approximation theorem, for each integer n there is a polynomial p_n such that

$$|f(x) - p_n(x)| < \epsilon \qquad \text{for } n-1 \leqq x \leqq n+1.$$

Define f_ϵ by

$$f_\epsilon(x) = \sum_{k\in Z} p_k(x)\cdot b(x-k);$$

then f_ϵ is infinitely differentiable for all x.

Consider $x=n$. Then $b(n-n)=1$ and $b(n-k)=0$ for any integer $k\neq n$. Hence $f_\epsilon(n)=p_n(n)$. Therefore $|f(n)-f_\epsilon(n)|<\epsilon$.

Consider $n<x<n+1$. Then $b(x-k)=0$ for all integers $k<n$ or $n+1<k$. Hence

$$f_\epsilon(x) = p_n(x)\cdot b(x-n) + p_{n+1}(x)\cdot b(x-n-1).$$

Now $-\epsilon<f(x)-p_n(x)<\epsilon$ and $-\epsilon<f(x)-p_{n+1}(x)<\epsilon$. Therefore

$$-(b(x-n)+b(x-n-1))\epsilon < (b(x-n)+b(x-n-1))f(x) - f_\epsilon(x) < (b(x-n)+b(x-n-1))\epsilon.$$

But $b(x-n)+b(x-n-1)=1$ for $n<x<n+1$. Therefore $|f(x)-f_\epsilon(x)|<\epsilon$. Since we have shown that $|f(x)-f_\epsilon(x)|<\epsilon$ for $n\leqq x<n+1$ and n is arbitrary, then $|f(x)-f_\epsilon(x)|<\epsilon$ for all x in R.

By using a more general partition of unity, our theorem and its method of proof can be extended to higher dimensional Euclidean spaces and to certain differentiable manifolds. By "differentiable" we mean "infinitely differentiable" and by "diffeomorphism" we will mean an "infinitely differentiable homeomorphism." Let X be the space under consideration. A *partition of unity* on X is a family $\{b_\alpha| : \alpha \text{ in } A\}$ of differentiable functions defined on X such that: (i) $0\leqq b_\alpha(x)\leqq 1$ for all α and all x; (ii) if $G_\alpha=\{x|b_\alpha(x)>0\}$, then the family of sets $\{G_\alpha|\alpha \text{ in } A\}$ is an open, locally finite cover of X (that is, each point x of X has an open neighborhood N_x which intersects only finitely many of the sets G_α); (iii) the closure of each G_α is diffeomorphic to a compact subset K_α of the Euclidean space R^n, $n=\dim X$; and (iv) $\sum_\alpha b_\alpha(x)=1$ for all x. Now given a continuous function f on X to find a differentiable function f_ϵ such that $|f(x)-f_\epsilon(x)|<\epsilon$ for all x, denote the diffeomorphism from K_α onto the closure of G_α by h_α and the appropriate polynomial approximation to $f\circ h_\alpha$ on K_α by p_α; then take

$$f_\epsilon = \sum_\alpha b_\alpha\cdot(p_\alpha \circ \overset{-1}{h_\alpha}).$$

Since the above was written we have discovered that our main theorem for the one variable case appears as a problem in *Introduction to Real Analysis* by Casper Goffman, Harper and Row, 1966, p. 99, problem 5.4.

The preparation of this paper was partially supported by NSF Research Grant GP-4604. The author is indebted to the referee for his suggestions for improving this paper.

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PLANAR INCIDENCE GEOMETRIES WITH TWO POINTS PER LINE

R. E. WHITNEY, Lock Haven State College

This paper discusses planar incidence geometries with two points per line. The axioms assumed may be found in [1, p. 121]. Formulae for calculating the number of fixed point automorphisms are derived. We first discuss notation.

A point, j, is fixed under a map, ϕ, if and only if $\phi(j)=j$. Let $E_k=\{1, 2, \cdots k\}$

Now $-\epsilon<f(x)-p_n(x)<\epsilon$ and $-\epsilon<f(x)-p_{n+1}(x)<\epsilon$. Therefore

$$-(b(x-n)+b(x-n-1))\epsilon<(b(x-n)+b(x-n-1))f(x)-f_\epsilon(x)$$
$$<(b(x-n)+b(x-n-1))\epsilon.$$

But $b(x-n)+b(x-n-1)=1$ for $n<x<n+1$. Therefore $|f(x)-f_\epsilon(x)|<\epsilon$. Since we have shown that $|f(x)-f_\epsilon(x)|<\epsilon$ for $n\leqq x<n+1$ and n is arbitrary, then $|f(x)-f_\epsilon(x)|<\epsilon$ for all x in R.

By using a more general partition of unity, our theorem and its method of proof can be extended to higher dimensional Euclidean spaces and to certain differentiable manifolds. By "differentiable" we mean "infinitely differentiable" and by "diffeomorphism" we will mean an "infinitely differentiable homeomorphism." Let X be the space under consideration. A *partition of unity* on X is a family $\{b_\alpha| : \alpha \text{ in } A\}$ of differentiable functions defined on X such that: (i) $0 \leqq b_\alpha(x)\leqq 1$ for all α and all x; (ii) if $G_\alpha=\{x|b_\alpha(x)>0\}$, then the family of sets $\{G_\alpha|\alpha \text{ in } A\}$ is an open, locally finite cover of X (that is, each point x of X has an open neighborhood N_x which intersects only finitely many of the sets G_α); (iii) the closure of each G_α is diffeomorphic to a compact subset K_α of the Euclidean space R^n, $n=\dim X$; and (iv) $\sum_\alpha b_\alpha(x)=1$ for all x. Now given a continuous function f on X to find a differentiable function f_ϵ such that $|f(x)-f_\epsilon(x)|<\epsilon$ for all x, denote the diffeomorphism from K_α onto the closure of G_α by h_α and the appropriate polynomial approximation to $f\circ h_\alpha$ on K_α by p_α; then take

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A point, j, is fixed under a map, ϕ, if and only if $\phi(j)=j$. Let $E_k=\{1,2,\cdots k\}$

$(k \geqq 1)$, $A_1 = \{x\}$, and $B_1 = \{y\}$ where $x \neq y$. Let $A_k = A_1 \cup E_{k-1}$ and $B_k = B_1 \cup E_{k-1}$ $(k > 1)$. We assume for any fixed k that $x, y \notin E_{k-1}$. We use the following combinatorial symbols:

$$\binom{k}{j} = \frac{k!}{(k-j)!j!} \; (0! \equiv 1) \quad \text{and} \quad P_j^k = k(k-1) \cdots (k-j+1), \; P_0^k \equiv 1.$$

Let m_k denote the number of (distinct) 1-1 correspondences from A_k to B_k with no fixed points and let n_j^k denote the number of permutations of E_k which leave exactly j $(0 \leqq j \leqq k)$ fixed points. Note that n_1^p represents the number of pseudo-rotations of G, the geometry. For example: $m_1 = 1$, $m_2 = 1$, $m_3 = 3$, $n_0^1 = 0$, $n_0^2 = 1$, $n_0^3 = 2$.

If the geometry, G, has p points, then the number of lines is $\binom{p}{2}$ and the total number of automorphisms is $p!$. Also if G is projective, then $p = 3$ and if G is affine, then $p = 4$ [1, p. 161]. If the set of points of G is E_p, then any permutation of E_p induces an automorphism of G since incidence is preserved.

Thus a combinatorial discussion of the permutations of E_p yields the number of fixed point automorphisms of G. Two recursive formulae for n_j^p will be derived.

THEOREM 1. $n_0^{k+1} = k m_k$ $(k \geqq 1)$.

Proof. Let ϕ be a permutation of E_{k+1} with no fixed points. $\phi(1) \in \{2, 3, \cdots, k+1\}$. Assume $\phi(1) = 2$. Then $\{2, 3, \cdots, k+1\}$ is mapped 1-1 onto $\{1, 3, \cdots, k+1\}$ with no fixed points and the result follows.

THEOREM 2. $m_k = (k-1)! + \sum_{j=0}^{k-3} P_j^{k-1} n_0^{k-j-1}$ $(k \geqq 3)$.

Proof (by induction). Let C_k denote the statement of the theorem. C_3 is true: $m_3 = 2 + n_0^2 = 3$. Assume C_t is true for some $t \geqq 3$. Let ϕ be a 1-1 correspondence from A_{t+1} to B_{t+1}.

Case 1. $\phi(x) = y$. The number of permutations of $E_t = A_{t+1} - \{x\} = B_{t+1} - \{y\}$ with no fixed points is n_0^t.

Case 2. $\phi(x) \neq y$, $\phi(x) \in \{1, 2, \cdots, t\} = E_t$. Assume $\phi(x) = 1$; then the number of 1-1 correspondences from E_t onto $\{y, 2, \cdots, t\}$ is m_t. Hence

$$m_{t+1} = n_0^t + t m_t = t! + \sum_{j=0}^{t-2} P_j^t n_0^{t-j}$$

by our induction hypothesis. Thus C_t implies C_{t+1} and the result follows by induction.

In the light of the above we have

THEOREM 3. (1) $n_0^p = (p-1)! + \sum_{j=1}^{p-3} P_j^{p-1} n_0^{p-j-1}$ $(p \geqq 4)$. n_j^p may be obtained from

THEOREM 4. (2) $n_j^p = \binom{p}{j} n_0^{p-j}$ $(0 \leqq j \leqq p$ *and* $p \geqq 2)$ *where* $n_0^0 \equiv 1$.

Proof. There are $\binom{p}{j}$ distinct sets of j fixed points, each of which leaves $p-j$ elements to be permuted with no fixed points. From (2) and the identity $\sum_{j=0}^{p} n_j^p = p!$, we obtain

THEOREM 5. (3) $n_0^p = p! - \sum_{j=1}^{p} \binom{p}{j} n_0^{p-j}$ $(p \geqq 1)$.

From (1), (2), and (3), one may calculate the number of fixed point automorphisms of G. Since the recurrence relations have variable coefficients, it is doubtful that more compact forms can be found.

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EXTENSION OF FEUERBACH'S FORMULA

FRANK DAPKUS, Seton Hall University

Feuerbach's Theorem establishes a well-known relationship between the radii of 4 circles tangent to 3 lines forming a triangle:

$$R_0^{-1} = R_1^{-1} + R_2^{-1} + R_3^{-1}$$

where R_0 is the radius of the circle inscribed in the triangle while the other 3 touch upon it externally. In this note we demonstrate an easy generalization of this formula to n dimensions.

In 3 dimensions let us consider 4 planes p_i defined by $A_ix+B_iy+C_iz+D_i=0$, $(i=1, 2, 3, 4)$ forming a tetrahedron. If $K_i = \pm\sqrt{(A_i^2+B_i^2+C_i^2)}$ the centers of the spheres tangent to all 4 planes are defined by the equations

$$K_i^{-1}(A_i\alpha + B_i\beta + C_i\gamma + D_i) = K_j^{-1}(A_j\alpha + B_j\beta + C_j\gamma + D_j)$$

for all pairs of i and j and for all sign combinations of K's; i.e., each center (α, β, γ) is obtained by solving a system of 3 linear equations by Cramer's Rule. Using the abbreviation $|a\ b\ c\ d|$ to designate a determinant with rows $a_i\ b_i\ c_i\ d_i$ $(i=1, 2, 3, 4)$ we can write the solutions as

$$\alpha = |K\ B\ C\ D|/|A\ B\ C\ K|,$$
$$\beta = |A\ K\ C\ D|/|A\ B\ C\ K|,$$
$$\gamma = |A\ B\ K\ D|/|A\ B\ C\ K|.$$

The radii of the spheres then can be computed by standard methods and are given by

$$R = |A\ B\ C\ D|/|A\ B\ C\ K|.$$

It is easy to see that the R's are determined solely by the sign combinations of K's. Our problem is now to relate the signs of K's to particular spheres.

Let us designate the sphere inscribed in the tetrahedron by S_0 and the four touching it externally as S_1, S_2, S_3, S_4 where S_i touches the tetrahedron on the side formed by p_i, the corresponding radii being R_i and the center coordinates $(\alpha_i, \beta_i, \gamma_i)$. We note that the centers of the spheres are intersection points of the bisector planes of the dihedral angles between the pairs of p_i. Every edge of the

THEOREM 5. (3) $n_0^p = p! - \sum_{j=1}^{p} \binom{p}{j} n_0^{p-j}$ $(p \geqq 1)$.

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Since R_0 is clearly the shortest of all radii, it must correspond to the sign combination of K's that maximizes $|ABCK|$, i.e., when the expansion of the determinant with respect to the fourth column yields all terms of equal sign. Now suppose we change the sign of some K_i. This changes all bisector planes where p_i is involved, i.e., they no longer pass through the interior of the tetrahedron and we get a new sphere touching the tetrahedron externally on the side p_i. Thus the radii R_1, R_2, R_3, R_4 differ from R_0 only by the sign of one K in the denominator of the formula defining the radii and this fact leads immediately to the 3 dimensional version of Feuerbach's Formula:

$$2\cdot R_0^{-1} = R_1^{-1} + R_2^{-1} + R_3^{-1} + R_4^{-1}.$$

The n-dimensional version of the formula can be obtained by defining a plane as a set of points $(x_1, \cdots, x_n)$ where $A_1x_1+ \cdots +A_nx_n+B=0$. Extending the point to plane distance formula and solving the system of n linear equations we obtain analogous expressions for the centers and the radii of the spheres tangent to all $n+1$ planes. For the radii we have

$$R = |A_1 \cdots A_nB| / |A_1 \cdots A_nK|.$$

where $K=\pm\sqrt{(A_1^2+ \cdots +A_n^2)}$.

If R_0 is considered as the smallest of these radii, and if the determinant in the denominator is expanded with respect to the last column, a generalization of Feuerbach's Formula to n dimensions follows immediately:

$$(n-1)R_0^{-1} = \sum_{i=1}^{n} R_i^{-1}.$$

CONFORMAL LINEAR TRANSFORMATIONS

ALI R. AMIR-MOÉZ, Texas Technological College

In this expository note we are more concerned about the idea of discovery of the theorem which characterizes a conformal linear transformation than in simply stating and proving the theorem.

1. Definitions and notations. A unitary space will be denoted by E. Vectors are denoted by Greek letters. The inner product of two vectors will be (ξ, ζ). The norm of ξ is $\|\xi\|=(\xi, \xi)^{1/2}$. We shall use standard notations for linear transformation and matrices [1]. The adjoint of A is A^* and is defined by $(A\xi, \zeta) = (\xi, A^*\zeta)$. A linear transformation A on E is called conformal if it satisfies tetrahedron is contained in two such bisector planes, one of which passes through the interior of the tetrahedron and the other one does not. It is easy to show that one can be obtained from the other by altering the sign of one K in the equation $K_i^{-1}(A_ix+B_iy+C_iz+D_i)=K_j^{-1}(A_jx+B_jy+C_jz+D_j)$ defining the bisector plane.

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$$\frac{(A\xi, A\zeta)}{\|A\xi\| \, \|A\zeta\|} = \frac{(\xi, \zeta)}{\|\xi\| \, \|\zeta\|}, \qquad \|A\xi\| \neq 0, \|A\zeta\| \neq 0, \tag{1}$$

for all nonzero ξ and ζ in E.

2. A matrix approach. Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal basis for E_n, an n-dimensional unitary space. We shall consider only matrices with respect to this basis. Since A satisfies (1) we must have $(A\alpha_i, A\alpha_j) = 0$ for $i \neq j$. Thus the matrix of AA^* will be

$$\begin{bmatrix} \|A\alpha_1\|^2 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \|A\alpha_n\|^2 \end{bmatrix}$$

We observe that $(A\xi, A\zeta) = (\xi, AA^*\zeta)$ and $\|A\xi\| = (\xi, AA^*\xi)^{1/2}$. Let $\xi = (x_1, \cdots, x_n)$ and $\zeta = (y_1, \cdots, y_n)$. Then (1) will be

$$\frac{\|A\alpha_1\|^2 x_1\bar{y}_1 + \cdots + \|A\alpha_n\|^2 x_n\bar{y}_n}{(\|A\alpha_1\|^2 |x_1|^2 + \cdots + \|A\alpha_n\|^2 |x_n|^2)^{1/2}(\|A\alpha_1\|^2 |y_1|^2 + \cdots + \|A\alpha_n\|^2 |y_n|^2)^{1/2}}$$
$$= \frac{x_1\bar{y}_1 + \cdots + x_n\bar{y}_n}{(|x_1|^2 + \cdots + |x_n|^2)^{1/2}(|y_1|^2 + \cdots + |y_n|^2)^{1/2}}.$$

Let some $x_i \neq 0$; for example suppose $x_1 \neq 0$. Since (1) is true for all $\xi, \eta \in E_n$ we choose $\zeta = (1, 0, \cdots, 0)$ and $\|\xi\| = 1$. Thus (1) will be

$$\frac{\|A\alpha_1\|}{(\|A\alpha_1\|^2 |x_1|^2 + \cdots + \|A\alpha_n\|^2 |x_n|^2)^{1/2}} = 1.$$

This implies that $\|A\alpha_1\| = \cdots = \|A\alpha_n\| \neq 0$. Therefore A is of the form of mU where m is a positive real number and U is unitary. Here

$$m = \frac{1}{\|A\alpha_i\|}, \quad i = 1, \cdots, n.$$

Since we can write $U = e^{i\theta}V$, where V is unitary, we have $A = pV = me^{i\theta}V$. So far a proposition for finite dimensional cases has been obtained. We shall try the theorem in general. In order to have a clearer view of the problem we state a theorem and its converse.

3. THEOREM. *Let c be a complex number and U be a unitary transformation on E. Then $A = cU$ is conformal.*

Proof. We observe that

$$\frac{(A\xi, A\zeta)}{\|A\xi\| \, \|A\zeta\|} = \frac{c\bar{c}(U\xi, U\zeta)}{|c| \, \|U\xi\| \, |c| \, \|U\zeta\|} = \frac{(\xi, \zeta)}{|\xi| \, |\zeta|}.$$

4. THEOREM (converse of 3). *If A is a conformal linear transformation on E, then $A = cV$, where c is a complex number and V is unitary.*

Proof. Without loss of generality we can choose $\|\xi\|=\|\zeta\|=1$. Thus (1) will become

$$\frac{(A\xi,\ A\zeta)}{\|A\xi\|\ \|A\zeta\|} = (\xi,\ \zeta),\ \|A\xi\| \neq 0,\ \|A\zeta\| \neq 0.$$

This equality implies that $(\xi,\ AA^*\zeta)=\|A\xi\|\ \|A\zeta\|(\xi,\ \zeta)$. Thus we can write $(\xi,\ [AA^*-\|A\xi\|\ \|A\zeta\|]\zeta)=0$, for all ξ and ζ with $\|\xi\|=\|\zeta\|=1$. Therefore $AA^*-\|A\xi\|\ \|A\zeta\|I=0$ which implies that

$$\frac{1}{\|A\xi\|\ \|A\zeta\|}AA^* = I,$$

for all ξ and ζ with $\|\xi\|=\|\zeta\|=1$. Let us choose $\xi=\zeta$. Then

$$\left(\frac{1}{\|A\xi\|}A\right)\left(\frac{1}{\|A\xi\|}A^*\right) = I.$$

Thus $B=(1/\|A\xi\|)A$ is unitary and we must have $(1/\|A\zeta\|)A^*=B^{-1}$ and $\|A\xi\|=\|A\zeta\|$, for all $\|\xi\|=\|\zeta\|=1$. Let $\|A\xi\|=1/(m)$, $m>0$. Then $(1/m)A=U$ is unitary and we can write

$$A = mU = me^{i\theta}V = cV$$

where V is unitary.

Reference

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A SEQUENCE-APPROACH TO UNIFORM CONTINUITY

JOHN H. STAIB, Drexel Institute of Technology

In recent years there has been a renewed interest in using sequence methods in the teaching of elementary calculus. Many proofs are, in this writer's opinion, made both more rigorous and more readable by such methods. In this note we shall consider an application of uniform continuity from the sequence point of view. (The proofs that follow will be presented here in the same amount of detail that I would use in the classroom. We take our functions to be real valued and our space to be Euclidean.)

DEFINITION 1. *Let* $\{P_n\}$ *and* $\{Q_n\}$ *be two point sequences that are related in the following manner*: $\{|P_n-Q_n|\}\to 0$. *Then we shall say that they are parallel sequences.*

THEOREM 1. *Let* $f(P)$ *be continuous in a closed, bounded set* S. *If* $\{P_n\}$ *and* $\{Q_n\}$ *are any two parallel sequences in* S, *then* $\{f(P_n)-f(Q_n)\}\to 0$.

Proof. Without loss of generality we can choose $\|\xi\| = \|\zeta\| = 1$. Thus (1) will become

$$\frac{(A\xi, A\zeta)}{\|A\xi\| \, \|A\zeta\|} = (\xi, \zeta), \|A\xi\| \neq 0, \|A\zeta\| \neq 0.$$

This equality implies that $(\xi, AA^*\zeta) = \|A\xi\| \, \|A\zeta\| (\xi, \zeta)$. Thus we can write $(\xi, [AA^* - \|A\xi\| \, \|A\zeta\|]\zeta) = 0$, for all ξ and ζ with $\|\xi\| = \|\zeta\| = 1$. Therefore $AA^* - \|A\xi\| \, \|A\zeta\| I = 0$ which implies that

$$\frac{1}{\|A\xi\| \, \|A\zeta\|} AA^* = I,$$

for all ξ and ζ with $\|\xi\| = \|\zeta\| = 1$. Let us choose $\xi = \zeta$. Then

$$\left(\frac{1}{\|A\xi\|} A\right)\left(\frac{1}{\|A\xi\|} A^*\right) = I.$$

Thus $B = (1/\|A\xi\|)A$ is unitary and we must have $(1/\|A\zeta\|)A^* = B^{-1}$ and $\|A\xi\| = \|A\zeta\|$, for all $\|\xi\| = \|\zeta\| = 1$. Let $\|A\xi\| = 1/(m)$, $m > 0$. Then $(1/m)A = U$ is unitary and we can write

$$A = mU = me^{i\theta}V = cV$$

where V is unitary.

Reference

1. A. R. Amir-Moéz and A. L. Fass, Elements of Linear Spaces, Pergamon Press, New York, 1962, pp. 123–130.

A SEQUENCE-APPROACH TO UNIFORM CONTINUITY

JOHN H. STAIB, Drexel Institute of Technology

In recent years there has been a renewed interest in using sequence methods in the teaching of elementary calculus. Many proofs are, in this writer's opinion, made both more rigorous and more readable by such methods. In this note we shall consider an application of uniform continuity from the sequence point of view. (The proofs that follow will be presented here in the same amount of detail that I would use in the classroom. We take our functions to be real valued and our space to be Euclidean.)

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THEOREM 1. *Let* $f(P)$ *be continuous in a closed, bounded set* S. *If* $\{P_n\}$ *and* $\{Q_n\}$ *are any two parallel sequences in* S, *then* $\{f(P_n) - f(Q_n)\} \to 0$.

Proof. (By contradiction.) Let us assume that $\{f(P_n)-f(Q_n)\}\not\to 0$. This means that there is an $\epsilon>0$ such that $|f(P_n)-f(Q_n)|>\epsilon$ for infinitely many n. Thus there exist subsequences $\{P_n'\}$ and $\{Q_n'\}$ such that $|f(P_n')-f(Q_n')|>\epsilon$ for all n.

Now, since S is a bounded set, the sequence $\{P_n'\}$ is bounded. Thus, by the Bolzano-Weierstrass Theorem, it has a convergent subsequence, say $\{P_n^*\}$. Let $\{P_n^*\}\to\alpha$. Because S is closed, we are assured that $\alpha\in S$. Because f is continuous in S, we are assured that $\{f(P_n^*)\}\to f(\alpha)$. But also, because $\{Q_n^*\}$ is parallel to $\{P_n^*\}$, we have $\{Q_n^*\}\to\alpha$. It follows that $\{f(Q_n^*)\}\to f(\alpha)$ and, consequently, that $\{f(P_n^*)-f(Q_n^*)\}\to 0$. But our opening assumption implies that $|f(P_n^*)-f(Q_n^*)|>\epsilon$ for all n.

THEOREM 2. *If f is continuous in the rectangle $\{(x, y): a\leqq x\leqq b, c\leqq y\leqq d\}$ and g is defined in $[a, b]$ by the formula*

$$g(x) = \int_c^d f(x, t)dt,$$

then g is continuous in $[a, b]$.

Proof. Let $\alpha\in[a, b]$ and suppose that $\{x_n\}$ is any α-approaching sequence in the domain of g. We wish to show that $\{g(x_n)\}\to g(\alpha)$. First, we write

$$g(x_n) = \int_c^d f(x_n, t)dt = \int_c^d [f(x_n, t) - f(\alpha, t) + f(\alpha, t)]dt$$

$$= \int_c^d [f(x_n, t) - f(\alpha, t)]dt + g(\alpha) = R_n + g(\alpha).$$

Thus, our original problem may be replaced by an equivalent one: we wish now to show that $\{R_n\}\to 0$.

We next observe that for each n the expression $f(x_n, t)-f(\alpha, t)$ defines a continuous function of t, say $h_n(t)$. Thus, the mean value theorem for integrals allows us to write

$$R_n = \int_c^d h_n(t)dt = (d-c)h_n(t_n) = (d-c)[f(x_n, t_n) - f(\alpha, t_n)],$$

where t_n is between c and d. Now the sequences $\{(x_n, t_n)\}$ and $\{(\alpha, t_n)\}$ are not, in general, convergent; however, they are parallel sequences lying in a closed rectangle. Thus, by Theorem 1, we are assured that $\{f(x_n, t_n)-f(\alpha, t_n)\}\to 0$. It follows that $\{R_n\}\to 0$.

Theorem 2 is usually proved by appealing to the uniform continuity of f in the given rectangle; here we have used instead a certain property of continuous functions with respect to parallel sequences. The notions are, as one might anticipate, equivalent. More precisely, we have the following sequence-definition for uniform continuity.

DEFINITION 2. *If for every pair of parallel point sequences* $\{P_n\}$ *and* $\{Q_n\}$ *taken from a set S we have* $\{f(P_n)-f(Q_n)\}\to 0$, *then f is said to be uniformly continuous in S.*

THEOREM 3. (Justification for Definition 2.) *If for every* $\epsilon>0$ *there is a* $\delta>0$ *such that* $|f(P)-f(Q)|<\epsilon$ *whenever P and Q are chosen from S in such a manner that* $|P-Q|<\delta$, *then f is uniformly continuous in S, and conversely.*

Proof. (1) Let $\epsilon>0$ be given and let δ be as described in the hypothesis. Now choose $\{P_n\}$ and $\{Q_n\}$ as any pair of parallel sequences in S. It follows that there is a number N such that

$$|P_n - Q_n| < \delta \quad \text{for } n > N.$$

But then, by our characterization of δ,

$$|f(P_n) - f(Q_n)| < \epsilon \quad \text{for } n > N.$$

Thus, $\{f(P_n)-f(Q_n)\}\to 0$ and, consequently, f is uniformly continuous in S.

(2) (By contradiction.) Let f be uniformly continuous in S but suppose that there is a troublesome ϵ for which no δ can be prescribed. Then, given a positive sequence $\{\delta_n\}$ that converges to 0, we can construct two sequences of points in the following manner: For each δ_n we choose points P_n and Q_n such that

$$\text{(a)} \quad |P_n - Q_n| < \delta_n \quad \text{and} \quad \text{(b)} \quad |f(P_n) - f(Q_n)| \geqq \epsilon.$$

By (a) we are assured that $\{P_n\}$ and $\{Q_n\}$ are parallel sequences. By (b) we are assured that $\{f(P_n)-f(Q_n)\}\not\to 0$. But this denies our assumption of uniform continuity!

We close with two examples showing how this sequence-definition of uniform continuity "works."

Example 1. To show that $f(x) = 1/(x^2+1)$ is uniformly continuous in $(-\infty, \infty)$, we must show that $\{f(x_n)-f(y_n)\}\to 0$, where $\{x_n\}$ and $\{y_n\}$ are an arbitrary pair of parallel sequences. First, we note that

$$|f(x_n) - f(y_n)| = \frac{|x_n^2 - y_n^2|}{(x_n^2+1)(y_n^2+1)} \leqq \frac{|x_n| + |y_n|}{r_n} \cdot |x_n - y_n|.$$

Now suppose that $|x_n|\leqq 1$ (for a particular n). Then

$$\frac{|x_n|}{r_n} \leqq \frac{|x_n|}{(1)(1)} \leqq 1.$$

And if $|x_n|>1$, then

$$\frac{|x_n|}{r_n} \leqq \frac{|x_n|}{(x_n^2)(1)} = \frac{1}{|x_n|} < 1.$$

Thus for all x_n (and y_n) we have $|x_n|/r_n\leqq 1$. Similarly, $|y_n|/r_n\leqq 1$. It follows that

$$|f(x_n) - f(y_n)| \leqq 2|x_n - y_n| \quad \text{for all } n.$$

But then, since $\{x_n - y_n\} \to 0$, we may conclude that $\{f(x_n) - f(y_n)\} \to 0$.

Example 2. To show that $f(x, y) = (x^2 y)/(1+y^4)$ is not uniformly continuous over the whole plane, take

$$P_n = (n, 1) \quad \text{and} \quad Q_n = (n + 1/n, 1).$$

Then $\{|P_n - Q_n|\} \to 0$, but

$$\{f(P_n) - f(Q_n)\} = \{(1/2)[n^2 - (n + 1/n)^2]\} = \{-1 - 1/(2n^2)\} \not\to 0.$$

A NOTE ON STEINER'S PROBLEM

P. N. BAJAJ, Case Western Reserve University

Let P be any point in the plane of an acute-angled triangle ABC. Steiner [1] proved, geometrically, that $PA + PB + PC$ is a minimum when each side of the triangle subtends an angle of $(2\pi)/3$ at P. S. M. Shah [2] derived an expression for the minimum distance in terms of the sides. S. Venkatramaiah [3] derived the same expression differently.

The purpose of the present note is to obtain Steiner's result using calculus methods.

Take (x_i, y_i), $i = 1, 2, 3$ for vertices of the triangle and P: (x, y); then the problem reduces to a discussion of $f(x, y) = \sum \sqrt{(x - x_i)^2 + (y - y_i)^2}$ for minima. Suppose AP, BP, CP make angles $\theta_1, \theta_2, \theta_3$ with the x axis.

$(\partial f)/(\partial x) = 0 = (\partial f)/(\partial y)$ gives

$$\sum \frac{(x - x_i)}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = 0 \quad \text{and} \quad \sum \frac{(y - y_i)}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = 0$$

or $\text{Cos } \theta_1 + \text{Cos } \theta_2 + \text{Cos } \theta_3 = 0$ and $\text{Sin } \theta_1 + \text{Sin } \theta_2 + \text{Sin } \theta_3 = 0$. This implies that

$$(\text{Cos } \theta_1 + \text{Cos } \theta_2)^2 + (\text{Sin } \theta_1 + \text{Sin } \theta_2)^2 = 1,$$

$$\text{Cos } (\theta_1 - \theta_2) = -1/2, \quad \text{and } |\theta_1 - \theta_2| = 2\pi/3.$$

Similarly, $|\theta_2 - \theta_3| = 2\pi/3$ and $|\theta_3 - \theta_1| = 2\pi/3$. The point obtained is one of minima as $f(x, y)$ can be made indefinitely large by taking (x, y) far enough from the triangle ABC.

References

1. R. Courant and H. Robbins, What is Mathematics?, Oxford University Press, New York 1941.

2. S. M. Shah, A Note on Steiner's Problem, Math. Student, 29 (1961).

3. S. Venkatramaiah, A remark on a note of S. M. Shah, this MAGAZINE, 39 (1966) 225.

$$|f(x_n) - f(y_n)| \leqq 2|x_n - y_n| \quad \text{for all } n.$$

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A SIMPLE PROOF OF AARON'S CONJECTURE ON THE FAREY SERIES

HWA S. HAHN, Pennsylvania State University

The Farey series F_n of order n is the ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed n. Thus y/x belongs to F_n if $0 \leqq y \leqq x \leqq n$, $(y, x) = 1$. For example, F_5 is 0/1, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1/1.

1. Blake [1] proved the following:

Aaron's Conjecture. The sum of the numerators of the fractions of a Farey series F_n is equal to one half the sum of the denominators of these fractions.

This note is to give a shorter proof. If y_i/x, $1 \leqq i \leqq k$, are the fractions in F_n with a fixed denominator x, the conjecture is even true for these fractions; namely, we shall show that

$$\sum_{i=1}^{k} y_i = \frac{1}{2} \sum_{i=1}^{k} x = kx/2$$

and the conjecture follows immediately from this. For $x = 1$ or 2 this is clear. For $x > 2$, if y/x belongs to F_n so does $(x-y)/x$ since $(y, x) = 1$ implies $(x-y, x) = 1$. And also $x - y \neq y$ for otherwise $y/x = 1/2$ and then $x = 2$. Thus by pairing off all y_i in $\sum y_i$ so that $y_i + (x - y_i) = x$ we obtain the above equality.

2. We make the following geometric observation of F_n. Consider distinct lines of the type $y = ax$ which pass through points (x, y) with integral coordinates, called lattice points, satisfying $0 \leqq y \leqq x \leqq n$. Then the sequence of slopes of these lines in ascending order is exactly F_n. From this observation Blake's second theorem follows.

THEOREM. *In F_n the denominator of the immediate predecessor and immediate successor of 1/2 is equal to the greatest odd integer $\leqq n$.*

If $n - 1 \leqq 2m + 1 \leqq n$, it is easy to see that both $m/(2m+1)$ and $(m+1)/(2m+1)$ are Farey fractions. It is now sufficient to show that inside the triangle bounded by $y = mx/(2m+1)$, $y = (m+1)x/(2m+1)$ and $x = n$ there is no lattice point except on $y = x/2$. But this is clear because the vertical distance from any lattice point, not on $y = x/2$, to $y = x/2$ is at least $\frac{1}{2}$ and the two points $(2m+1, m)$ and $(2m+1, m+1)$ are the only points in the triangle, including the boundary, which have the vertical distance $\frac{1}{2}$.

Reference

1. J. A. Blake, Some characteristic properties of the Farey series, Amer. Math. Monthly, 73 (1966) 50–52.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114

Algebra Through Problem Solving. By Abraham P. Hillman and Gerald L. Alexanderson. Series—Topics in Contemporary Mathematics. Allyn and Bacon, Boston, 1966. \$2.95 (paper).

This little volume is a neat collection of problems divided into ten chapters. Each chapter contains a brief introduction to the theme of the problems in that chapter. A look at the chapter headings reveals: 1. The Pascal Triangle; 2. The Fibonacci and Lucas Numbers; 3. Factorials; 4. Arithmetic and Geometric Progressions; 5. Mathematical Induction; 6. The Binomial Theorem; 7. Combinations and Permutations; 8. Polynomial Equations; 9. Determinants; 10. Inequalities.

Just what makes this book outstanding among all such books on an equal footing which the reviewer has seen? First, this book " . . . is an outgrowth of the authors' work in conducting problem solving seminars for undergraduates and high school teachers, . . . " among other things, to quote from the authors' preface. Second, (again quoting them) "Knowledge of mathematics together with the ability to apply this knowledge to nonroutine problems will be very valuable in the more and more automated world we face. Routine problems will tend to be solved mechanically, while new and challenging problems arise at a rapid rate for human minds to solve." Perhaps many problem books owe their presence in the printed world to such statements. And, if so, then indeed they are worthwhile. This is true of the present problem book. Certainly the authors' quoted statements are borne out throughout their book, and many of the problems are definitely nonroutine, but only so with forceful, careful anticipation of them.

Problems in each chapter fall nicely into patterns of increasing generality which gives the problem solver patterns of inference (we were about to say, "patterns of plausible inference.") This forceful and careful arranging of the problems is the main merit of the book. These outstanding features of the book work in harmony. Being able to solve nonroutine problems in algebra gives mathematical knowledge (among other things). And being forcefully and carefully led up to nonroutine problems may well bring out this ability.

R. M. VOGT, San Jose State College

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While the table of contents lists some expected topics such as prime numbers, congruences, irrationals and iterations, Diophantine equations, continued fractions and Fibonacci numbers, the treatment of these topics is always fresh and vibrant, and sometimes novel. For example, that $x^3+y^3=z^3$ has no solution in positive integers is, of course, well known, but a discussion which uses the graph of $x^3+y^3=1$ and talks of everywhere dense sets breathes new life into the topic, even for a more advanced mathematician. Besides developing formulae for Pythagorean triples, the authors show that the inscribed circle of the 3-4-5 triangle has radius 1 and answer the question, "How many noncollinear points in a plane can be spaced at integral distances each from each?" When perfect numbers are discussed, the authors give a delightful little proof that all perfect numbers, except 6, have a "digital sum" of 1, but 6 is an exception because 2 is the only even prime.

That a question quite easy to ask can in fact lead one far afield in quest of an answer is aptly illustrated by the following problem: "What is the probability that two numbers selected at random are relatively prime?," which leads to a discussion of probabilities, convergence of infinite series, multiplication of infinite products, and the Riemann-Zeta-function.

As well as a thorough discussion of prime numbers, several unusual ideas regarding primes are given, such as that some of the properties of primes may be consequences of the sieving process rather than of primality. In fact, some of our unsolved mathematical problems may be because we haven't yet asked the right questions.

The spirit of mathematical inquiry as expressed in this book would be enough reason to encourage mathematics students to read it. Students and teachers alike could benefit from the book as background reading. The book is accurate, attractively written, virtually self-contained, and up-to-date.

MARJORIE R. BICKNELL, A. C. Wilcox High School, Santa Clara, California

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This remarkable little book should be placed high upon the "must have" list of every secondary and college mathematics teacher. The authors have been completely successful in introducing the subject of topology in a self-contained, elementary, rigorous and readable package. Furthermore, the reader who has studied this book will come away with the "correct" idea of what topology and topologists are doing. In short, the authors present a topologist's view of topology.

The development centers upon two existence theorems which give sufficient conditions for the existence of a solution to the equation $fx=y$ when f is a continuous map of either (i) a closed interval into R^1 or (ii) a closed disk into R^2. All of the topological concepts required in the statement and proof of these results, and there are many such concepts, are introduced. As illustrative consequences there are such corollaries as the Brouwer Fixed Point Theorem, the ham sandwich theorem and the fundamental theorem of algebra.

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The preceding brief outline does not begin to describe the wealth of detail, the motivating discussions, the well-chosen exercises of this book. All in all, a better introduction to topology for the layman would be difficult to imagine.

J. G. HOCKING, Michigan State University

Calculus Part One. By Morris Kline. Wiley, New York, 1967. xiii+574 pp. $9.95.

It is refreshing to find that books are still being written on elementary calculus in which the subject is approached from an intuitive and heuristic standpoint rather than attempting to inculcate in the beginning student the deadly rigorous viewpoint of the sophisticated mathematician.

Applications are introduced almost from the beginning. These add interest to the subject for the student. It is not until the seventh chapter that the chain rule is discussed and proven. A fairly complete treatment of translation and rotation of axes including fundamental invariants is given. The inverse trigonometric functions are handled more fully than is customary. Definite integrals are postponed to Chapter 14. The approach is slow with many geometric illustrations and heuristic arguments.

It does seem misleading to use degree measure on graphs of trigonometric functions. On the top of page 235 the identity should read $\sin x/2 = \pm\sqrt{(1-\cos x/2)}$, since the negative sign must be used if $x/2$ is an angle in the third or fourth quadrants.

R. W. COWAN, Lamar State College of Technology

Calculus Part Two. By Morris Kline. Wiley, New York, 1967. xiii+415 pp. $8.75.

The second part of Kline's Calculus is similar to the first part. A number of applications are found including motion of a projectile, velocity and acceleration in curvilinear motion, and Kepler's laws. The approach to infinite series is very gradual beginning with a polynomial approximation to functions and Taylor's Formula with a remainder. An introduction to solid analytic geometry precedes partial differentiation and multiple integration. This book concludes with two chapters devoted to a reconsideration of the foundations of the subject. This treatment is not postulational but it does provide a more complete insight into the concept of a function, limit of a function and of a sequence, and the definite integral. Some of this material could be inserted early in the course or be deferred until much later depending on the class and the inclination of the instructor.

For the purpose for which they were designed these calculus books by Morris Kline would be splendid texts for a beginning student who needs a book that is readable, interesting, and not too demanding in mathematical rigor.

R. W. COWAN, Lamar State College of Technology

The preceding brief outline does not begin to describe the wealth of detail, the motivating discussions, the well-chosen exercises of this book. All in all, a better introduction to topology for the layman would be difficult to imagine.

J. G. HOCKING, Michigan State University

Calculus Part One. By Morris Kline. Wiley, New York, 1967. xiii+574 pp. $9.95.

It is refreshing to find that books are still being written on elementary calculus in which the subject is approached from an intuitive and heuristic standpoint rather than attempting to inculcate in the beginning student the deadly rigorous viewpoint of the sophisticated mathematician.

Applications are introduced almost from the beginning. These add interest to the subject for the student. It is not until the seventh chapter that the chain rule is discussed and proven. A fairly complete treatment of translation and rotation of axes including fundamental invariants is given. The inverse trigonometric functions are handled more fully than is customary. Definite integrals are postponed to Chapter 14. The approach is slow with many geometric illustrations and heuristic arguments.

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R. W. COWAN, Lamar State College of Technology

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in india ink and exactly the size desired for reproduction.

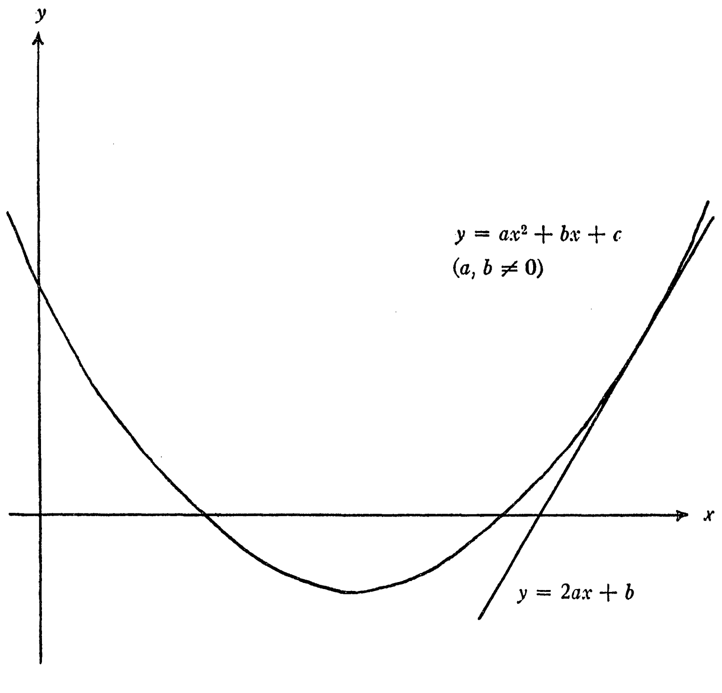
Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

To be considered for publication, solutions should be mailed before February 1, 1968.

PROBLEMS

669. *Proposed by Robert P. Baker, Newark, New Jersey.*

Why is the figure below impossible?



670. *Proposed by Maxey Brooke, Sweeny, Texas.*

Find the unique solution:

· · · nothing you dismay!

$$\begin{array}{r} MERRY \\ XMAS \\ FROM \\ \hline MAXEY \end{array}$$

671. *Proposed by A. Wilansky, Lehigh University.*

Let $t_1, t_2, \cdots, t_r$ be real numbers with $t_1+t_2+\cdots+t_r=0$ and let $\{x_n\}$ be a bounded sequence of real numbers. Show that if the $\lim_{n\to\infty}(t_1x_{n-r+1}+t_2x_{n-r+2}+\cdots+t_rx_n)$ exists, it must be zero.

672. *Proposed by R. S. Luthar, Colby College, Maine.*

Prove the theorem: *A sufficient condition that the equation $x^2-dy^2=-I$ be soluble in integers is that d be of the form $I+n^2$ or $5+4n(n+I)$ where n is any integer.*

673. *Proposed by Erwin Just, Bronx Community College.*

If n is an integer such that

$$\begin{bmatrix} k & m \\ 0 & k+1 \end{bmatrix}^n = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \qquad \text{prove that } a_2 = m[(k+1)^n - k^n].$$

674. *Proposed by J. A. H. Hunter, Toronto, Canada.*

Max Rumney (London, England) recently conjectured that $6^n/2\pm1$ must generate at least one prime for all values of n. Prove or disprove this conjecture.

675. *Proposed by Philip Fung, Cleveland State University, Ohio.*

It is well known that in an obtuse triangle, the minimum number of lines necessary for partitioning into acute triangles is seven. Show a constructional method for such a partition.

676. *Proposed by William Squire, West Virginia University.*

Evaluate $\sum_{k=1}^{\infty} A_k^3$ where A_k is the factorial quotient

$$\frac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots2k}.$$

SOLUTIONS

Late Solutions

David C. Hoaglin: **643**; *Erland M. H. Polden and Malcolm A. Perella, Rutherford College of Technology, Newcastle Upon Tyne, England:* **647** (*Jointly*); *Stanley Rabinowitz, Far Rockaway, New York:* **647, 648.**

An Alphametic

649. [March, 1967] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm

$$\begin{array}{r} THREE \\ +\ \ FOUR \\ \hline SEVEN \end{array}$$

in the decimal system such that:

3 does not divide $THREE$ in which the digit 3 is missing;
4 does not divide $FOUR$ in which the digit 4 is missing;
7 does not divide $SEVEN$ in which the digit 7 is missing.

Solution by Harry M. Gehman, SUNY at Buffalo, New York.

Let us first solve the cryptarithm, given only that
(a) the digit 3 is missing from $THREE$;
(b) the digit 4 is missing from $FOUR$;
(c) the digit 7 is missing from $SEVEN$.
The problem has seven solutions:

(1)	16544 7805 ——— 24349	(2)	47266 9102 ——— 56368
(3)	75244 9102 ——— 84346	(4)	79244 5102 ——— 84346
(5)	17544 6805 ——— 24349	(6)	49266 7102 ——— 56368
(7)	24811 6708 ——— 31519		

The condition (d) that 3 does not divide $THREE$ eliminates solutions (5) and (6). The condition (e) that 4 does not divide $FOUR$ eliminates (7). The condition (f) that 7 does not divide $SEVEN$ does not eliminate any solution.

Therefore the problem as proposed has four solutions: (1)–(4).

If we ignore conditions (d) (e) (f) but retain conditions (a) (b) (c) with the additional condition indicated we have unique solutions as follows:

(g) $THREE$ contains the digit 8. Solution (7).
(h) $SEVEN$ contains the digit 1. Solution (7).
(i) $FOUR$ contains both the digits 5 and 6. Solution (5).
(j) $THREE$ contains neither 6 nor 7. Solution (7).
(k) $THREE$ contains both 6 and 7. Solution (2).
(l) $THREE$ contains both 1 and 2. Solution (7).
(m) $THREE$ contains neither 5, 6 nor 7. Solution (7).
(n) $THREE$ contains neither 5, 7 nor 9. Solution (7).

and so on.

The fact that solution (7) occurs so frequently in this list seems to indicate that it has a pattern of digits essentially different from the other six solutions. From the standpoint of numerology, this has some deep significance, I am sure.

Complete solutions also submitted by John Beidler, Scranton University, Pennsylvania; Pierre Bouchard, Université de Montréal, Canada; and Charles W. Trigg, San Diego, California.

Partial solutions by M. J. Adler, Willowdale, Ontario, Canada; Merrill Barnebey, Wisconsin State University at LaCrosse; Maxey Brooke, Sweeny, Texas; Sarah Brooks, Utica Free Academy, New York; Helen Edens, Richmond, Virginia; Daniel Fettner, City College of New York; Regina R. Hoelscher; J. A. H. Hunter, Toronto, Ontario, Canada; Richard A. Jacobson, Houghton College, New York; Edgar Karst, University of Arizona; Fred Lambie, Lexington, Massachusetts; Sam Newman, Atlantic City, New Jersey; Lawrence V. Novak, University Park, Pennsylvania; Arlene Peterson, Immaculata College, Pennsylvania; Marilyn R. Rodeen, Balboa High School, San Francisco, California; Michael J. Sheridan, San Diego, California; John H. Tiner, Harrisburg, Arkansas; James H. Turner, New Wilmington, Pennsylvania; and the proposer.

Homothetic Triangles

650. [March, 1967] *Proposed by Charles W. Trigg, San Diego, California.*

At a distance from each side of the triangle ABC equal to the length of that side and on the vertex side of that side, a line is drawn parallel to that side. These three lines determine a triangle $A'B'C'$ similar to ABC. Show that $A'A$, $B'B$ and $C'C$ are concurrent at a point P whose distances from the sides of ABC are proportional to the sides.

Solution by Gregory Wulczyn, Bucknell University, Pennsylvania.

Since triangles ABC and $A'B'C'$ are by the hypothesis perspective from the line at infinity, they are by the converse of Desargues' Theorem perspective from a point P. That is AA', BB' and CC' are concurrent at P.

As the axis of perspectivity is the line at infinity, the perspectivity is an homothecy (similarity transformation) with homothetic center at P.

Let P_{BC} be the distance from P to BC, etc., then

$$\begin{aligned} k &= P_{B'C'}/P_{BC} = P_{C'A'}/P_{CA} = P_{A'B'}/P_{AB} \\ &= (P_{B'C'} + P_{BC})/P_{BC} = (P_{C'A'} + P_{CA})/P_{CA} \\ &= (P_{A'B'} + P_{AB})/P_{AB}. \end{aligned}$$

Thus we have $k=a/P_{BC}=b/P_{CA}=c/P_{AB}$ or $P_{BC}:P_{CA}:P_{AB}=a:b:c$.

Also solved by Pierre Bouchard, Université de Montréal, Canada; Huseyin Demir, Middle East Technical University, Ankara, Turkey; Michael Goldberg, Washington, D.C.; Lew Kowarski, Morgan State College, Maryland; Stanley Rabinowitz, Far Rockaway, New York; and the proposer.

Demir and Trigg pointed out that P is the Lemoine point of the triangles, and the concurrency of the symmedians of a triangle has been shown in a manner different from the usual one.

Probability of Real Roots

651. [March, 1967] *Proposed by Frank Dapkus, Seton Hall University.*

Let $x^2+bx+c=0$ have roots x_0 such that $|x_0|\leqq a$. What is the probability that x_0 is real?

Solution by Michael Goldberg, Washington, D.C.

It is assumed that all pairs of real values b, c are equally probable. Then, each equation is represented by a point in the b, c plane. Since $x_0=(-b\pm\sqrt{b^2-4c})/2$, the real roots exist only for the region in which $b^2 \geqq 4c$, while the imaginary roots exist only in the region for which $b^2 < 4c$. These regions are separated by the parabola whose equation is $b^2 = 4c$.

The region in which $|x_0| = a$ is a closed region containing the origin. The top of this region is the horizontal straight line $c = a^2$ in the imaginary region since then, $x_0 = (-b + i\sqrt{4c - b^2})/2$, and $|x_0| = \{b^2 + (4c - b^2)\}/4 = c = a^2$. Hence $|x_0| = a$. The line meets the parabola at the points $b = \pm\sqrt{4a^2} = \pm 2a$. The slope of the tangent at $b = 2a$ is $b/2 = a$. The equation of the tangent line is $b = 2a - (a^2 - c)/a$ or $a^2 - ab + c = 0$. Hence, $a = (b \pm \sqrt{b^2 - 4c})/2$ and $|x_0| = a$. Hence, this tangent line is also a boundary of the region in which $|x_0| \leqq a$. The other tangent line symmetric about the c-axis is another boundary. The enclosed area in which $|x_0| \leqq a$ is $a(2a)^2 = 4a^3$.

The part of the area within the parabola is $8a^3/3$. Hence, the real part of the area is $4a^3/3$, and the probability of a real root is $1/3$, regardless of the value of a.

Also solved by Richard A. Jacobson, Houghton College, New York; James R. Kuttler, Vincent G. Sigillito, and James T. Stadter, Silver Springs, Maryland (Jointly); Charles W. Trigg, San Diego, California; Stephen Weintraub, Oceanside, New York; Raymond E. Whitney, Lock Haven State College, Pennsylvania; and the proposer.

Merrill Barnebey, Wisconsin State University at LaCrosse pointed out that the problem was discussed at an NSF Summer Institute at Rutgers University where different results were obtained depending upon different assumptions regarding the space.

Pairs of Twin Primes

652. [March, 1967] *Proposed by Merrill Barnebey, Wisconsin State University at LaCrosse.*

Prove that the sum of any pair of twin primes greater than seven is divisible by twelve.

I. *Solution by Sister Marion Beiter, Rosary Hill College, New York.*

The problem may be generalized: Let p and q be two odd integers such that $q = p + 2$ and neither p nor q is divisible by three. Then p, $p+1$, q are three consecutive numbers, of which $p+1$ is divisible by three and is an even number. The sum, $p + q = 2(p+1)$, is divisible by 3 and by $2 \cdot 2$, hence by twelve.

II. *Comment by Vassili Daiev, Sea Cliff, New York.*

In general, if the integers $2m+1$ and $2m-1$ are not divisible by 3 then their sum is divisible by 12. These numbers can be: (1) both prime, (2) one prime and the other composite, (3) both composite. (Examples: 29 and 31; 47 and 49; 143 and 145). Such pairs are of the form $6n-1$ and $6n+1$. It is evident that there are infinitely many pairs of such composite numbers.

Also solved by Leon Bankoff, Los Angeles, California; Merrill Barnebey, Wisconsin State University at LaCrosse; Gladwin E. Bartel, Whitworth College, Washington; Donald Batman, M.I.T. Lincoln Laboratory; John Beidler, Scranton University, Pennsylvania; Murray Berg, San Francisco, California; Pierre Bouchard, Université de Montréal, Canada; Mary Beth Bridgham, Northwestern University; Maxey Brooke, Sweeny, Texas; Mannis Charosh, Brooklyn, New York; Mickey Dargitz, Ferris State College, Michigan; James A. Darragh, California State College at Long Beach; Phillip F. Dean, Suitland, Maryland; Huseyin Demir, Middle East Technical University, Ankara, Turkey;

Monte Dernham, San Francisco, California; Mary Ellien, Pratt Institute, New York; William F. Feeney, University of Pittsburgh; David Fettner, City College of New York; Frank Fishell, Montcalm Community College, Michigan; John Flavin; Larry Forman, Brown University; Philip Fung, Cleveland State University, Ohio; Stephen M. Gagola, Jr., Buffalo, New York; Harry M. Gehman, SUNY at Buffalo, New York; Reinaldo Giudici, University of Pittsburgh; Anton Glaser, Pennsylvania State University, Abington, Pennsylvania; Michael Goldberg, Washington, D.C.; Sandra A. Gossum, University of Tennessee; Robert E. Harper, Eastern Kentucky University; Ned Harrell, Los Altos, California; Harry W. Hickey; John E. Homer, Jr., Lisle, Illinois; J. A. H. Hunter, Toronto, Ontario, Canada; Richard A. Jacobson, Houghton College, New York; Douglas H. Johnson, University of Wisconsin; Allan W. Johnson, Jr., Towson, Maryland; Erwin Just, Bronx Community College, New York; Edgar Karst, University of Arizona; Lew Kowarski, Morgan State College, Maryland; Sidney Kravitz, Dover, New Jersey; James R. Kuttler, Vincent M. Sigillito and James T. Stadter, Silver Springs, Maryland (Jointly); J. F. Leetch, Bowling Green State University, Ohio; Herbert R. Leifer, Pittsburgh, Pennsylvania; Douglas Lind, University of Virginia; Michael J. Martino, Temple University, Pennsylvania; Sam Newman, Atlantic City, New Jersey; H. Panish, Northrop Nortronics Company, California; Bud Pass, Georgia Institute of Technology; Donna E. Pitz, Northern Illinois University; Bob Prielipp, University of Wisconsin; Stanley Rabinowitz, Far Rockaway, New York; Kenneth A. Ribet, Brown University; Marilyn R. Rodeen, Balboa High School, San Francisco, California; Stephen Spindler, Purdue University; Daniel R. Stark, Cleveland State University, Ohio; E. P. Starke, Plainfield, New Jersey; John H. Tiner, Harrisburg, Arkansas; James Turner, New Wilmington, Pennsylvania; Zalman Usiskin, University of Michigan; John Wessner, Melbourne High School, Florida; Mrs. Raymond E. Whitney, Lock Haven State College, Pennsylvania; Gregory Wulczyn, Bucknell University, Pennsylvania; Charles Ziegenfus, Madison College, Virginia; and the proposer. One unsigned solution was received.

Exponential Derivative

653. [March, 1967] *Proposed by by Sam Newman, Atlantic City, New Jersey.*

What is dy/dx of

$$y = x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}} \left.\vphantom{x^{x^{x^{x}}}}\right\} n \qquad ?$$

I. *Solution by John Beidler, Scranton University, Pennsylvania.*

For any nonnegative integer n, denote y by

$$y = f(x, n)$$

where $f(x, 0) = x$. Also denote (dy/dx) by $f'(x, n)$. A formula for the production of (dy/dx) may be produced by taking, for $n > 0$, the identity

$$\ln[f(x, n)] = f(x, n - 1) \ln x$$

and differentiating both sides with respect to x; then solving for $f'(x, n)$ yields the formula

$$f'(x, n) = f(x, n)[x^{-1}f(x, n - 1) + f'(x, n - 1) \ln x].$$

This along with the fact that $f'(x, 0) = 1$ will generate the desired result for any positive integral value of n.

$$\text{If } n = 1 \qquad \frac{dy}{dx} = x^x[1 + \ln x]$$

$$n = 2 \qquad \frac{dy}{dx} = x^{x^x} x^x \left[\frac{1}{x} + \ln x + (\ln x)^2\right]$$

$$n = 3 \qquad \frac{dy}{dx} = x^{x^{x^x}} x^{x^x}\left[\frac{1}{x} + x^x\left(\frac{\ln x}{x} + (\ln x)^2 + (\ln x)^3\right]\right.$$

etc.

II. *Solution by Stanley Rabinowitz, Far Rockaway, New York.*

Let f_n denote the function $x^{x^{\cdot^{\cdot^{x}}}}$ (where there are nx's). $(df_n/dx) = f_nf_{n-1}/x + f_n(\ln x)df_{n-1}/dx$. Using this formula, one easily finds that

$$\frac{df_n}{dx} = f_nf_{n-1}/x + f_nf_{n-1}f_{n-2}(\ln x)/x + f_nf_{n-1}(\ln x)^2\frac{df_{n-2}}{dx}.$$

Continuing to substitute, one gets by induction

$$\frac{df_n}{dx} = \sum_{j=1}^{k}\left[\frac{(\ln x)^{j-1}}{x}\prod_{i=0}^{j} f_{n-i}\right] + \left[\prod_{i=0}^{k-1} f_{n-i}\right](\ln x)^k\frac{df_{n-k}}{dx}.$$

When $k = n-1$, we have

$$\frac{df_n}{dx} = \sum_{j=1}^{n-1}\left[\frac{(\ln x)^{j-1}}{x}\prod_{i=0}^{j} f_{n-i}\right] + (\ln x)^{n-1}\prod_{i=0}^{n-2} f_{n-i}$$

III. *Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

The given function may be defined by the recurrence relation $y_n = x^{y_{n-1}}$, $y_1 = x^x$, $y_0 = x$, $y_{-1} = 1$. Taking logarithms and differentiating we obtain

$$\frac{y_n'}{y_n} = \frac{y_{n-1}'}{y_{n-1}}(y_{n-1}\ln x) + \frac{y_{n-1}}{x}$$

Writing the last equality from $n = 1$ up to $n = n$ and multiplying each relating by a suitable factor and adding them up we get

$$y_n' = \sum_{k=0}^{n} y_ny_{n-1}\cdots y_{n-k-1}(\ln x)^k/x.$$

Also solved by Pierre Bouchard, Université de Montréal, Canada; Nicholas C. Bystrom, St. Paul, Minnesota; Richard W. Feldman, Lycoming College, Pennsylvania; David Fettner, City College of New York; Reinaldo E. Giudici, University of Pittsburgh; Michael Goldberg, Washington, D.C.; Sandra A. Gossum, University of Tennessee; J. M. Howell, Los Angeles City College; Richard A. Jacobson, Houghton College, New York; Lew Kowarski, Morgan State College, Maryland; Fred Lambie, Lexington, Massachusetts; Douglas Lind, University of Virginia; Edwin A. Power, University College, London, England; and the proposer. A number of incorrect or undecipherable solutions were received.

Ten Kids' Fortunes

654. [March, 1967] *Proposed by Sidney Kravitz, Dover, New Jersey.*

Ten kids own a total of 2,879 pennies. The ratio of every kid's fortune to the fortune of each kid poorer than himself is an integer. If no two kids own the same amount, find the fortune of each kid.

I. *Solution by Philip Fung, Cleveland State University, Ohio.*

Let $x_i=\prod_{j=1}^{i} n_j$ such that $x_{i-1}<x_i$, x_j divides x_i for all $j<i$ and $\sum_{i=1}^{10} x_i=2879$. Since 2879 is a prime it follows that $n_1=1$ and $n_j\geqq 2$ for $j>1$.

Define $S(j)=1+\sum_{l=j}^{10}\prod_{k=j}^{l} n_k$, $j\geqq 2$. Clearly $S(j)-1=n_jS(j-1)$, whence we obtain $n_i=2$ for $i=2, 3, 4, 5, 6, 9, 10$ and $n_7=4$, $n_8=3$. Thus the distribution of pennies becomes: $\{1, 2, 4, 8, 16, 32, 128, 384, 768, \text{ and } 1536\}$.

II. *Solution and comments by Joseph S. Madachy, Kettering, Ohio.*

Let the first kid's fortune be A; the second kid's fortune, AB; the third, ABC; the fourth, $ABCD$; and so on to the tenth fortune, $ABCDEFGHIJ$. Then

$$A + AB + ABC + ABCD + ABCDE + ABCDEF + ABCDEFG + ABCDEFGH + ABCDEFGHI + ABCDEFGHIJ = 2879$$

or

$$A(1 + B + BC + BCD + \cdots + BCDEFGHIJ) = 2879.$$

Since 2879 is prime it has only two factors 1 and 2879, so $A=1$ and

$$B(1 + C + CD + CDE + \cdots + CDEFGHIJ) = 2878 = (2)(1439).$$

1439 is also prime, and so $B=2$. (Since no kids own the same amount, no other fortune can be in the ratio of 1 to any other, except A.) This leads to

$$C(1 + D + DE + DEF + \cdots + DEFGHIJ) = 1438 = (2)(719)$$

719 is prime and so $C=2$. Similarly, $D=2$, $E=2$, and $F=2$. Eventually we reach

$$G(1 + H + HI + HIJ) = 88 = (2)(44) = (4)(22) = (8)(11).$$

If $G=2$, 8, 11, 22, or 44 we would reach the conclusion that either H or I or J equal 1, which would contradict the condition of uniqueness of fortunes. However, if $G=4$, then

$$H(1 + I + IJ) = 21 = (3)(7).$$

If $H=7$, then either I or J equal $1-a$ contradiction to conditions. Therefore, $H=3$, yielding $I=2$ and $J=2$.

The ten kids have, respectively, the following number of pennies:

1, 2, 4, 8, 16, 32, 128, 384, 768, and 1536

making a total of 2879 pennies.

The past history of this problem (see the entry for 7 kids in the table below) suggested the testing of the number of possible solutions for n kids (n from 1 to ?). The solutions found are given in the table below: (A completely general solution would use m pennies—I leave that work to someone else!)

Number of kids	Solutions (Pennies for each kid)									
1	2879									
2	1,	2878								
3	1,	2,	2876							
4	1,	2,	4,	2872						
5	1,	2,	4,	8,	2864					
6	1,	2,	4,	8,	16,	2848				
7*	1,	2,	4,	8,	16,	32,	2816			
8	1,	2,	4,	8,	16,	32,	64,	2752		
	1,	2,	4,	8,	16,	32,	128,	2688		
	1,	2,	4,	8,	16,	32,	256,	2560		
	1,	2,	4,	8,	16,	32,	352,	2464		
	1,	2,	4,	8,	16,	32,	704,	2112		
9	1,	2,	4,	8,	16,	32,	128,	384,	2304	
	1,	2,	4,	8,	16,	32,	128,	896,	1792	
	1,	2,	4,	8,	16,	32,	256,	512,	2048	
10	1,	2,	4,	8,	16,	32,	128,	384,	768,	1536

No solutions for more then 10 kids.

* This variation (7 kids) of the problem was proposed by Kravitz in the December, 1962, issue (No. 12) of *Recreational Mathematics Magazine* (Page 20). The 7-kid variation was also repeated in "Mathematics On Vacation" by Joseph S. Madachy (Charles Scribner's, 1966) on Page 120.

Also solved by Leon Bankoff, Los Angeles, California; Merrill Barnebey, Wisconsin State University at LaCrosse; Donald Batman, MIT Lincoln Laboratory; John Beidler, Scranton University; Murray Berg, San Francisco, California; C. Berndtson, Kwajalein, Marshal Islands; Arthur Bolder, Brooklyn, New York; Pierre Bouchard, Université de Montréal, Canada; Robert J. Bridgman, Mansfield State College, Pennsylvania; Maxey Brooke, Sweeny, Texas; Sarah Brooks, Utica Free Academy, New York; Donald R. Chand and Sham S. Kapur, Lockheed Georgia Company (Jointly); Huseyin Demir, Middle East Technical University, Ankara, Turkey; David Fettner, City College of New York; Harry M. Gehman, SUNY at Buffalo, New York; Michael Goldberg, Washington, D.C.; Robert E. Harper, Eastern Kentucky University; J. M. Howell, Los Angeles City College; J. A. H. Hunter, Toronto, Ontario, Canada; Richard A. Jacobson, Houghton College, New York; Douglas H. Johnson, University of Wisconsin; Allan W. Johnson, Jr., Towson, Maryland; Edgar Karst, University of Arizona; Lew Kowarski, Morgan State College, Maryland; Sam Kravitz, East Cleveland, Ohio; James R. Kuttler, Vincent M. Sigillito and James T. Stadter, Johns Hopkins University (Jointly); Herbert R. Leifer; Douglas Lind, University of Virginia; Michael J. Martino, Temple University; Donald A. Myers, University of Wyoming; Sam Newman, Atlantic City, New Jersey; Harry Panish, Northrop Nortronics, Anaheim, California; Stanley Rabinowitz, Far Rockaway, New York; Kenneth A. Ribet, Brown University; Michael J. Sheridan, San Diego, California; Stephen Spindler, Purdue University; E. P. Starke, Plainfield, New Jersey; John H. Tiner, Harrisburg, Arkansas; Charles W. Trigg, San Diego, California; Zalman Usiskin, Ann Arbor, Michigan; Gregory Wulczyn, Bucknell University, Pennsylvania; and the proposer.

Usiskin pointed out that if one kid left his fortune to the others, the given conditions can be trivially satisfied by considering the number 2879 in base 2, giving each kid one of the powers of 2 in the base 2 expansion.

Homothetic Figures

655. [March, 1967] *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

It is easy to show that any two spheres are homothetic, regardless of their orientation. Show that this property characterizes spheres; that is, if two bounded figures are homothetic, regardless of their orientation, then they both must be spheres.

Solution by Pierre Bouchard, Université de Montréal, Canada.

It is easy to show that $\{x: |x| \in (1, 2), x \in R^3\}$ and $\{x: |x| \in (3, 6), x \in R^3\}$ are not spheres and are homothetic, regardless of their orientation. This negates the proposal as stated. However, we can prove that the given figures must have a frontier which is the union of a set S of concentric spheres, the cardinality of S being the same in each figure. But "seen" from "outside the bounds" they look like spheres. We proceed to prove this last fact.

Let F_1 and F_2 be the "exterior frontiers" of the given figures in a given position: more precisely $F_i = \{x: \exists y$ such that $|y| = 1$ and $x = \sup z_{z=cy z \in \mathfrak{F}_i}\}$ where $i = 1, 2$ and $\mathfrak{F}_i$ is the ith figure $c \in R$. First remark that for every y on the unit sphere there is a corresponding x (because of the "regardless of orientation"; otherwise the figures would be unbounded or void). Since $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are homothetic regardless of orientation, so are F_1 and F_2 (since affine homothety is translation or central homothety we may restrict ourselves to central homothety).

Let P_1, P_2 be in F_1. Then there is an α in R such that αP_1, αP_2 are in F_2. Let r be a rotation such that $r(\alpha P_1)$ is on the line OP_2 and $r(\alpha P_2)$ is on the line OP_1 (i.e., r is the rotation of π with respect to the axis passing through 0 and $\frac{1}{2}(P_1/|P_1| + P_2/|P_2|)$). Then F_1 and $r(F_2)$ are homothetic and since $\sup z_{z \in \mathfrak{F}}$, $z = cy$ is unique we must have $r(\alpha, P_1) = \beta P_2$, $r(\alpha P_2) = \beta P_1$. Since r preserves distances,

$$\begin{aligned} \beta &= |r(\alpha P_1)| / |P_2| \\ &= |\alpha(P_1)| / |P_2| \\ &= \alpha |P_1| / |P_2| \end{aligned}$$

and

$$\begin{aligned} \beta &= |r(\alpha P_2)| / |P_1| \\ &= |\alpha(P_2)| / |P_1| \\ &= \alpha |P_2| / |P_1| \end{aligned}$$

Whence $|P_1|/|P_2| = |P_2|/|P_1|$, or $|P_2| = |P_1|$. That is, F_1 is a sphere so is F_2 (homothetic image of a sphere).

Also solved by Michael Goldberg, Washington, D.C.; J. F. Leetch, Bowling Green University, Ohio; and the proposer.

Klamkin suggested that the counterexample exhibited by Bouchard could be eliminated by adding to the statement of the problem the qualifying statement, "bounded closed convex figures." Both "also solvers" tacitly made such an assumption in their solutions.

Comment on Problem 636

636. [November, 1966, and May, 1967] *Proposed by Vassili Daiev, Sea Cliff, New York.*

The greatest divisors of the form 2^k of the numbers of the sequence 2, 4, 6, 8, 10, 12, 14, $\cdots$ are 2, 2^2, 2, 2^3, 2, 2^2, 2, $\cdots$. Find the nth term of this sequence.

Comment by Kenneth A. Ribet, Brown University.

The solution by Michael Goldberg lends significance to the number of zeros to the right of the last "1" in the binary expansion of a number n. I would like to point out that we can attach significance as well to the total number of 1's appearing in the binary expansion of n.

Let n be given and suppose that p is a prime number. Then the exponent of the highest power of p that divides $n!$ is the sum

$$\sum_{k=1}^{\infty} [n/p^k]$$

If we write n in base p as $\alpha_0+\alpha_1 p+ \cdots +\alpha_r p^k$, then the sum becomes

$$\frac{1}{p-1}\left(n-\sum_{k=0}^{r}\alpha_k\right)$$

In particular, if $p=2$ and if n is the binary number $b_0b_1 \cdots b_s$, we see immediately that the highest power of two dividing $n!$ is two to the exponent

$$n-\sum_{k=0}^{s} b_k$$

In other words, the sum is the difference between n and the exponent of the highest power of two dividing n!

Example: $12=(1100)_2$. Therefore the highest power of two that divides 12! is $2^{12-2}=2^{10}$.

Errata

In A338, September, 1966, Page 226, the last line should read

$$\int_{\sqrt{2}}^{\infty}\frac{dx}{x+x^{\sqrt{2}}}=(\sqrt{2}+1)\log[1+2^{(1-\sqrt{2})/2}]$$

On Page 168, May, 1967, the last syllable should be "pro" not "pre."

On Page 170, May, 1967, in Q411, the word "odd" should be inserted after "two."

In A410, May, 1967, Page 140, the expression for x_i^{-1} should read

$$x_i^{-1} = \left[\prod_{\substack{1 \leq j \leq n \\ j \neq p \\ j \neq i}} x_j^2 \right] x_i x_p$$

for $1 \leqq i \leqq n$, $i \neq p$

$$x_p^{-1} = \left[\prod_{\substack{1 \leq j \leq n \\ j \neq p}} x_j^2 \right]$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q416. Determine the range of the function $I(t)$ where

$$I(t) = \int_0^\infty \frac{dx}{(x^2+1)(x^t+1)}.$$

[*Submitted by Murray S. Klamkin*]

Q417. If R is the circumradius and r the inradius of a triangle, show that $r/R+R/r \geqq 5/2$ with equality only when the triangle is equilateral.

[*Submitted by Leon Bankoff*]

Q418. Given two primes separated by $2k-1$ integers with both primes greater than $2k+1$. What is the g.c.d. for the product of all the integers between any two such primes?

[*Submitted by Brother Alfred Brousseau*]

Q419. Do there exist bounded sets S_1 and S_2 in E^n such that for every Euclidean motion γ, $\gamma(S_1) \cap S_2 \neq \varnothing$ implies $\gamma(S_1) \cap S_2$ is not convex?

[*Submitted by James P. Burling*]

(Answers on page 254)

Clearly f is a function of Baire class one, but it is not almost continuous. (See Example 3.)

This example also shows that the pointwise limit of a sequence of almost continuous functions is not necessarily almost continuous.

(G) *An almost continuous function on* $[a, b]$ *need not be bounded.*

Example 5. Let $\{E_n\}$ be a sequence of pairwise disjoint sets in $[0, 1]$ such that each E_n is dense everywhere in $[0, 1]$ and $\cup E_n = [0, 1]$. It is not hard to construct such sets. Define $f(x) = n$ if $x \in E_n$ where $n = 1, 2, \cdots$. Then f is almost continuous but unbounded.

The author is thankful to Professor W. Orlicz for this example.

This research was supported by the National Research Council of Canada and a McMaster University research grant.

Added in proof: Theorem 1 appeared recently in the Canad. Bull. of Math.

References

1. E. W. Hobson, The Theory of Functions of a Real Variable, vol. 1, Dover, New York, 1957.
2. T. Husain, Almost continuous mappings, Prace Mat., Series 1, 10 (1965) 1–7.
3. ———, $B(T)$-spaces and the closed graph theorem, Math. Ann., 153 (1954) 293–298.
4. W. Rudin, Principles of Mathematical Analysis, McGraw-Hill, New York, 1953.

Answers

A416.

$$I(t) = \int_0^1 \frac{dx}{(x^2+1)(x^t+1)} + \int_1^\infty \frac{dy}{(y^2+1)(y^t+1)}.$$

In the second integral let $y = 1/x$. We then obtain

$$I(t) = \int_0^1 \frac{dx}{x^2+1} = \pi/4.$$

Thus the range of $I(t) = 0$. This integral appears in *Induction and Analogy in Mathematics*, by G. Polya.

A417. The relation can be verified by substituting $2r + x$ for R. When the triangle is equilateral, $x = 0$ and $R = 2r$.

A418. Including the two primes there are $2k+1$ consecutive integers which have a g.c.d. of $(2k+1)!$ Since the primes do not enter into this g.c.d., it must be the g.c.d. for the numbers between the primes.

A419. Put $S_1 = \{(P_1, P_2, \cdots, P_n) \mid 0 \leq P_i \leq 1, P_i \text{ rational}\}$, and

$$S_2 = \{(x_1, x_2, \cdots, x_n) \mid 2 < x < 3, 0 < x_1 < 1 \text{ for } i > 1\}.$$

These sets exhibit the characteristic indicated in the question.

(Quickies on page 289)

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